1. Lecture 1

Many physical systems are governed by physical laws involving rates of change (e.g. Newton's second law of motion states that the force upon an object is equal to mass times acceleration, $F = ma$). Since rates of change are just derivatives, the types of equations that we want to study are called differential equations.

If a differential equation models a physical, biological or other "real-world" situation, we call that model a **mathematical model**.

Question: What can be modeled by differential equations?

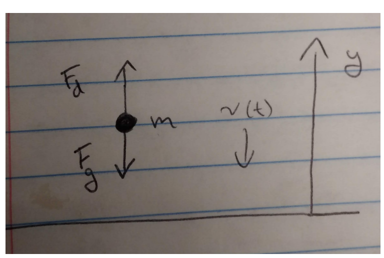
A: Movement (falling objects), interest rate calculations, predator/prey relationships, the dynamics of springs and even things like the blink synchronization of fireflies.

Example 1.1. Suppose you drop an object from a large height, and want to model the velocity of the object in free fall. You're told (neglecting some physics technicalities) that the force of drag is proportional to the objects velocity. What is the differential equation governing this velocity?

***Solution*:** Let us define some things.

$$\begin{aligned} y(t) &= \text{position above the ground of the object at time } t \\ v(t) &= \frac{dy}{dt} = \text{velocity of the object at time } t \\ a(t) &= \frac{dv}{dt} = \frac{d^2y}{dt^2} = \text{acceleration of the object at time } t \end{aligned}$$

We draw a free-body diagram (here v points down because the object in falling, positive velocity still points upwards):



Since y is position above the ground, $F_{\text{gravity}} = -mg$. The minus sign represents that this force is in the opposite direction of "up."

Since the force of drag is proportional to the velocity we can write that $F_{\text{drag}} = -\gamma v$, where γ is some number. Here the minus sign represents that the force of drag is in the opposite direction of velocity.

Adding the forces together give

$$F_{\text{total}} = F_{\text{gravity}} + F_{\text{drag}} = -mg - \gamma v.$$

Since $F = ma$ (Newton's second law), and $a = \frac{dv}{dt}$ we get the differential equation:

$$m\frac{dv}{dt} = -mg - \gamma v.$$

□

Example 1.2. Suppose there's some field where the population of mice increases by 50% every month if there were no predators. However, there's a group of owls that kill 10 mice per day. If t is measured in months (assumed to be 30 days), what is the differential equation governing the population, p, of mice.

***Solution*:** We first ignore the owls. Let's gather some data:

January	February
100	150
200	300
300	450

.

If we carry this out more, we'll see that if p is the population in January, then $1.5p = p + \frac{1}{2}p$ is the population in February. Therefore, the $\frac{dp}{dt} = \frac{1}{2}p$ (again this is neglecting the owls).

We now take into effect the owls. The owls kill 300 mice per month and so the total differential equation is:

$$\frac{dp}{dt} = \underbrace{\frac{1}{2}p}_{\text{birth of mice}} - \underbrace{300}_{\text{killing of mice}}.$$

In general if the mice increased in population by $(100r)\%$ and mice kill k mice per month then the differential equation is

$$\frac{dp}{dt} = rp - k.$$

□

We now move onto solving some of these differential equations. We have two different ODEs:

$$\frac{dv}{dt} = -g - \frac{\gamma}{m}v \qquad \text{and} \qquad \frac{dp}{dt} = rp - k.$$

These are both of the form:

$$\frac{dy}{dt} = ay - b,$$

where a and b are numbers.

Let's completely solve the mice problem. We have

$$\frac{dp}{dt} = \frac{1}{2}p - 300 = \frac{1}{2}(p - 600).$$

This is called a *separable* differential equation. Dividing both sides by $p-600$ and multiplying both sides by dt we get

$$\frac{dp}{p - 600} = \frac{1}{2}dt.$$

We now integrate both sides:

$$\ln|p - 600| = \int \frac{dp}{p - 300} = \int \frac{dt}{2} = \frac{1}{2}t + C.$$

We can exponentiate both sides to undo the natural logarithm

$$|p - 600| = e^{\frac{1}{2}t+C} = e^C e^{t/2}.$$

We can get rid of the absolute value by adding a $\pm$ on the right and side. That is:

$$p - 600 = \pm e^C e^{t/2} = Ae^{t/2},$$

where $A = \pm e^C$ (and cannot be 0).

Adding 600 gives

$$p(t) = 600 + Ae^{t/2},$$

which is the general solution.

2. Lecture 2

How can we determine the constant A? Well we need an initial value! For example, if we said $p(0) = 500$ then we can say that

$$500 = p(0) = 600 + Ae^{0/2} = 600 + A,$$

or $A = -100$.

This tells us that the population *decreases* over time.

Combining what we have done in the previous computations we have solved the **initial value problem** (IVP):

$$\text{IVP:} \qquad \begin{cases} \frac{dp}{dt} &= \frac{1}{2}p - 300 \\ p(0) &= 500 \end{cases} .$$

Let's go back to the general form of the mice population/velocity differential equations and solve an initial value prolbem there. That is we want to solve:

$$\text{IVP:} \qquad \begin{cases} \frac{dy}{dt} &= ay - b \\ y(0) &= y_0 \end{cases} ,$$

where y_0, a and b are some numbers (and we will describe restrictions on them later).

We first solve the differential equation part of the IVP. That is we want to solve:

$$\begin{aligned}
\frac{dy}{dt} &= ay - b = a\left(y - \frac{b}{a}\right). \\
\frac{dy}{y - \frac{b}{a}} &= a\,dt \\
\int \frac{dy}{y - \frac{b}{a}} &= \int a\,dt \\
\ln\left|y - \frac{b}{a}\right| &= at + C \\
\left|y - \frac{b}{a}\right| &= e^{at+C} = e^C e^{at} \\
y - \frac{b}{a} &= \pm e^C e^{at} = Ae^{at} \\
y &= \frac{b}{a} + Ae^{at}.
\end{aligned}$$

This is the general solution to the differential equation at the top of the string of equalities.

We now have to find out A, and we use the initial value to compute A. That is, we have

$$y_0 = y(0) = \frac{b}{a} + Ae^{a\cdot 0} = \frac{b}{a} + A.$$

Therefore, $A = y_0 - \frac{b}{a}$. Thus

$$y(t) = \frac{b}{a} + \left[y_0 - \frac{b}{a}\right] e^{at}, \tag{1}$$

is the **general solution** to the initial value problem:

$$\text{IVP:} \qquad \begin{cases} \frac{dy}{dt} &= ay - b \\ y(0) &= y_0 \end{cases} .$$

Before moving on, we know that A cannot be 0. What does this mean in our situation? That means that $y_0 \neq \frac{b}{a}$. Another problem is when $a = 0$. I'll leave it to you to solve the IVP:

$$\text{IVP:} \qquad \begin{cases} \frac{dy}{dt} &= -b \\ y(0) &= y_0 \end{cases} .$$

Example 2.1. Use (1) to solve the initial value problem:

$$\text{IVP:} \qquad \begin{cases} \frac{dv}{dt} &= -g - \frac{\gamma}{m}v \\ v(0) &= v_0 \end{cases} .$$

***Solution*:** We have $a = -\gamma/m$, $b = g$ and $y_0 = v_0$. Plugging that in gives:

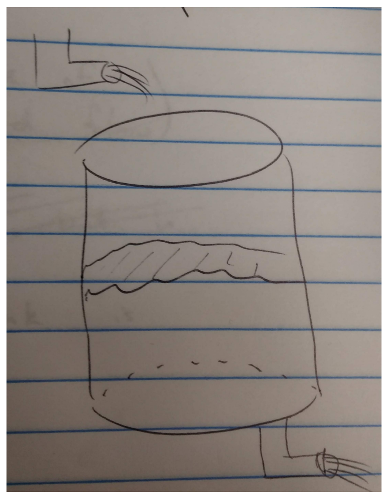
$$v(t) = -\frac{mg}{\gamma} + \left(v_0 + \frac{mg}{\gamma}\right) e^{-\gamma t/m}.$$

□

3. Lecture 3

Let's do a concrete example.

Example 3.1. A tank contains 100 L of water and 10 kg of salt and is mixed thoroughly. A pipe pours pure water into the tank, where it mixes instantaneously. To keep the tank from overflowing, there's another pipe at the bottom that drains the water from the tank at the same rate at which the water is pumped in. Suppose that both the pipes flow water into or out of the tank at a rate of 5 L per minute. Write a differential equation which governs the quantity measured in kilograms (i.e. the mass), q, of the salt in the tank at time t measured in minutes. Here's a (very poor) drawing.



***Solution*:** In words we can write:

$$\frac{dq}{dt} = \begin{pmatrix} \text{rate of} \\ \text{salt in} \end{pmatrix} - \begin{pmatrix} \text{rate of} \\ \text{salt out} \end{pmatrix}.$$

Since only pure water comes into the tank, the rate of salt in is 0.

The rate of the salt out can be compute by

$$\begin{pmatrix} \text{rate of} \\ \text{salt in} \end{pmatrix} = \begin{pmatrix} \text{density of} \\ \text{salt in the tank} \end{pmatrix} \times \begin{pmatrix} \text{flow rate} \\ \text{of water out} \\ \text{of the tank} \end{pmatrix}.$$

Since density if mass/volume, and q is the mass of the salt in the 100L tank we get the density of salt in the tank is $q/100$ with units kg/L. The flow rate is 5L per minute and so:

$$\begin{pmatrix} \text{rate of} \\ \text{salt in} \end{pmatrix} = \frac{q \text{ kg}}{100 \text{ L}} \times \frac{5 \text{ L}}{1 \text{ min}} = \frac{q}{20} \frac{\text{kg}}{\text{min}}.$$

Thus the IVP we must solve is

$$\begin{cases} \frac{dq}{dt} & = -\frac{1}{20}q \\ q(0) & = 10. \end{cases}$$

I'll leave it to you to use (1) to write down the exact solution. □

We now state some definitions (and elaborate from what was done in lecture):

A **differential equation** is just some equation relating the derivatives $y', y'', \ldots$, the function itself y to the time variable t (or sometimes the space variable x).

For example these are differential equations:

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^2 - 4y = e^x$$

and

$$y^{(5)} - y' = \sin(x).$$

Remark 3.1. There are several different notations for derivatives.

$$\begin{aligned} \frac{d^n y}{dx^n} &= \text{ the } n^{\text{th}} \text{ derivative of } y \text{ with respect to } x. \\ y', y'', y^{(n)} &= \text{ respectively, the first, second and } n^{\text{th}} \text{ derivative of } y. \\ \dot{y}, \ddot{y} &= \text{ respectively, the first and second derivative of } y \\ &\quad \text{(generally with respect to time).} \end{aligned}$$

The **order** of a differential equation is the order of the highest derivative which is included in the equation. For example if we write a differential equation as

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0,$$

for some function F then the differential equation is of order n.

A differential equation is **linear** if it can be written in the form:

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = g(x),$$

where $a_0, a_1, \ldots, a_n$ and g are functions involving only x. In fact, that differential equation is an n^{th} order linear differential equations.

Example 3.2. Which of the following are linear differential equations? If they are not, then why not? What is the order of the differential equation? Write the differential equations in the form:

$$F(x, y, y', \dots, y^{(n)}) = 0.$$

a) $y'' + \sin(x)y = 6$
b) $(y')^2 - y^2 = 0$
c) $e^t y + e^y y' = \ln(t)$
d) $y'' + 6y - 6y^{(3)} = 0$
e) $y'' + 2y' - 4y = \sin(y)$

***Solution*:**

a) This is a linear second order differential equation and
$$F(x, y, y') = y'' + \sin(x)y - 6.$$
b) This is a non-linear first order differential equation and
$$F(x, y, y') = (y')^2 - y^2.$$
It is non-linear because we square both the y' term and y term.
c) This is a non-linear first order differential equation and
$$F(x, y, y') = e^t y + e^y y' - \ln(t).$$
It is non-linear because there is a term involving e^y.
d) This is a linear 3rd order differential equation and
$$F(x, y, y', y'', y^{(3)}) = y'' + 6y - 6y^{(3)}.$$
e) This is a non-linear differential equation of order 2 and
$$F(x, y, y', y'') = -\sin(y) + y'' + 2y' - 4y.$$

□

Back to definitions (just 1 more).

A **separable differential equation** is a first order differential equation if we can write

$$\frac{dy}{dt} = f(t)g(y)$$

for some functions f and g.

For example,

$$\frac{dy}{dt} = y^2 te^{3t+4y} = te^{3t} \cdot y^2 e^{4y}$$

is a separable differential equation.

Example 3.3. Solve the separable differential equation:

$$e^{2x}y\frac{dy}{dx} = e^y + e^{y-2x}.$$

***Solution*:** We first have to write this in the form

$$\frac{dy}{dx} = f(x)g(y).$$

To get there we start by factoring the RHS:

$$e^y + e^{y-2x} = e^y(1 + e^{-2x}).$$

We can then divide everything by ye^{2x} to get:

$$\frac{dy}{dx} = \frac{e^y}{y} \cdot (e^{-2x} + e^{-4x}).$$

We now solve this by seperating variables:

$$ye^{-y}\, dy = \left(e^{-2x} + e^{-4x}\right)\, dt.$$

Integrating the LHS requires using integration by parts:

$$\int ye^{-y}\, dy = -ye^{-y} - \int -e^{-y} dy = -ye^{-y} - e^{-y},$$

where we used $u = y$ and $dv = e^{-y}\, dy$ and the IBP formula

$$\int u\, dv = uv - \int v\, du.$$

The RHS is easier, and becomes:

$$\int \left(e^{-2x} + e^{-4x}\right)\, dx = -\frac{1}{2}e^{-2x} - \frac{1}{4}e^{-4x} + C.$$

There is no constant of integration on the LHS because we put the constant of integration on the RHS.

Combining these into one equation we get:

$$-ye^{-y} - e^{-y} = -\frac{1}{2}e^{-2x} - \frac{1}{4}e^{-4x} + C,$$

this cannot be solved for y in a nice way so we'll simplify as much as we can:

$$\begin{aligned} -ye^{-y} - e^{-y} &= -\frac{1}{2}e^{-2x} - \frac{1}{4}e^{-4x} + C \\ -(y+1)e^{-y} &= \frac{-1}{4}\left(2e^{-2x} + e^{-4x} + C\right) \qquad \text{different } C \\ 4(y+1)e^{-y} &= 2e^{-2x} + e^{-4x} + C. \end{aligned}$$

□

We now outline a general method to solve the seperable equation:

$$\frac{dy}{dt} = f(t)g(y).$$

Step 1: Divide both sides by $g(y)$ and multiply both sides by dt to get:

$$\frac{dy}{g(y)} = f(t)\,dt.$$

Step 2: Integrate both sides, putting the constant of integration only on the side including the t terms:

$$\int \frac{dy}{g(y)} = \int f(t)\,dt + C.$$

Step 3: If possible, solve for y. If you cannot do that then write the equation in implicit form.

4. Lecture 4

Last time we covered separable equations, those of the form

$$\frac{dy}{dt} = f(t)g(y).$$

Those are nice in the sense that we have a general algorithm to solve the differential equation. The solutions may not be in the nicest form (i.e. we have to leave the equation in a weird implicit form instead of explicitly solving for y), but at least we can write down some equation that works.

Under certain conditions (which we'll mention later) you can show that a solution exists and is unique, but it is still hard to write down an actual solution. For example, currently we do not know how to solve the differential equation:

$$2\frac{dy}{dt} = 6 - 4t - y.$$

We'll learn how to solve this, but in the mean time you can check that $y(t) = Ae^{-2t} + 14 - 4t$ is a solution.

In the mean time we want to find some characteristic properties of what the solution to that differential equation may be. There are (roughly speaking) two ways we can get some properties of the solutions to first order differential equations when an explicit solution is not known:

- Graphically using these things called **direction fields** or **slope fields**.
- Numerically using something called **Euler's method**. There are more advanced methods which are called Runge-Kutta methods which will not be covered in this course.

Let's start by doing them graphically. Here is the direction field for the differential equation

$$y' = 3 - 2t - \frac{1}{2}y:$$

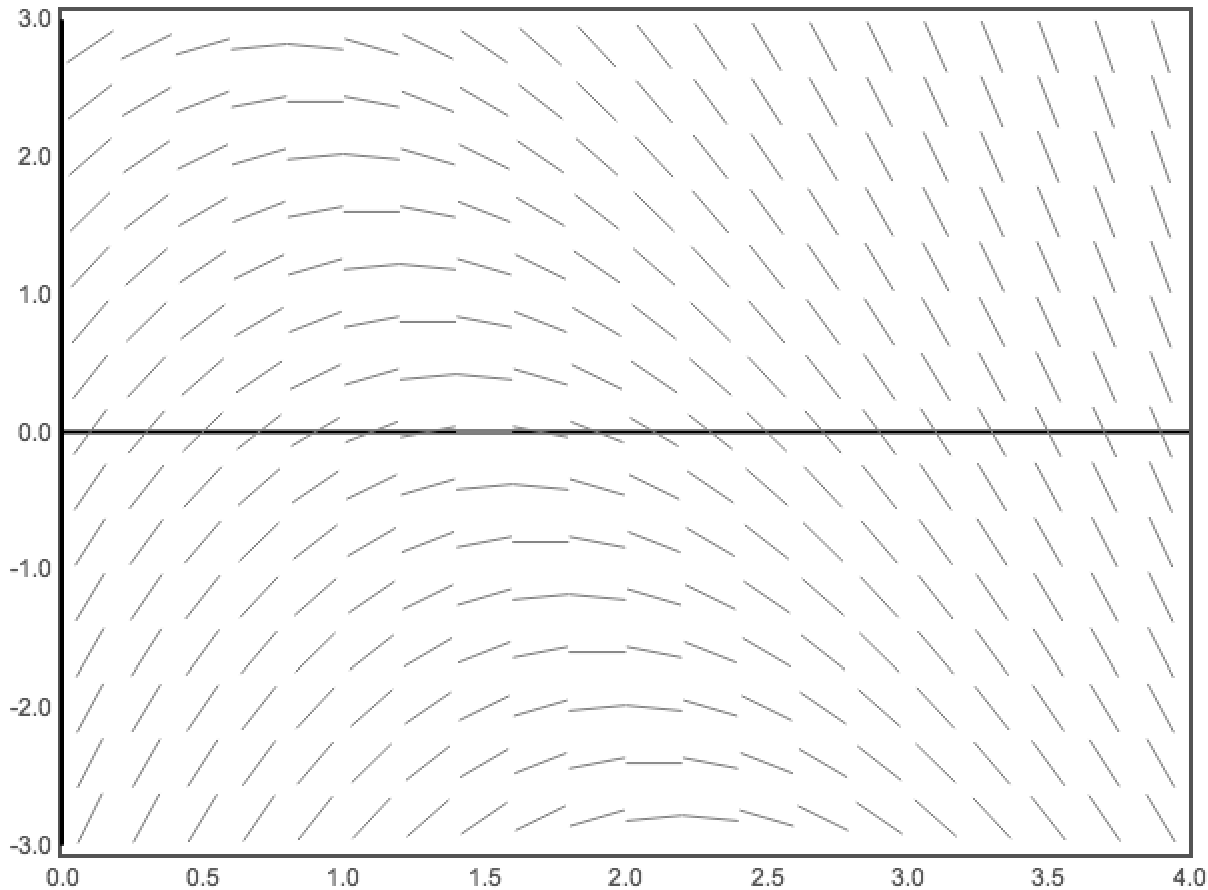


FIGURE 1. Slope field for $y' = 3 - 2t - \frac{1}{2}y$.

Of course, computers are much better at this than humans are, which will be more evident below, when I include a hand-drawn example. Professor Darryl Nester at Bluffton University's website has a great online slope-field grapher which is available at `https://bluffton.edu/homepages/facstaff/nesterd/java/slopefields.html`

Remark 4.1. I will <u>not</u> ask you to draw a slope field from scratch with as much detail as detail as the above example. If one is asked, it will just be to get the idea of what's happening with a solution, i.e. when is y' positive, negative or zero or if y' does or does not depend on t.

In general how do we draw a slope field for a general first order differential equation, say $y' = f(t, y)$? There are roughly 3 steps:

Step 1: Draw your axis and mark units in an appropriate manner.
Step 2: Mark grid points at which you will draw a small line segments.
Step 3: At a grid point (t_0, y_0), make a small line segment with slope $f(t_0, y_0)$.

You can see below my slope field for the differential equation $\frac{dp}{dt} = \frac{1}{2}p - 300$.

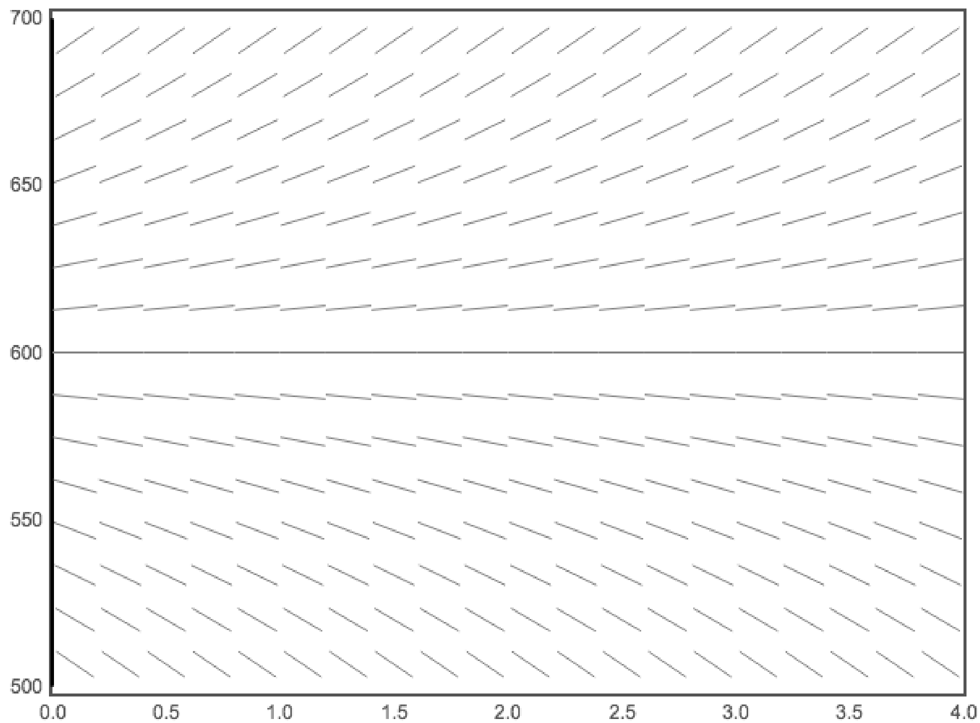


FIGURE 2. Computer slope field for $p' = \frac{1}{2}p - 300$.

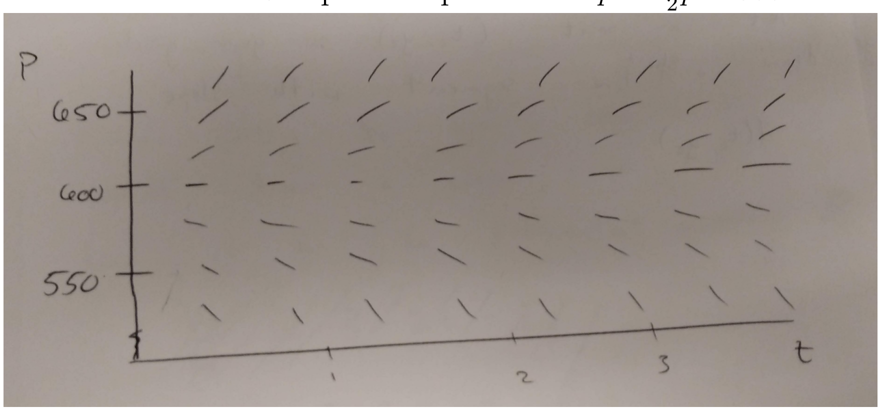


FIGURE 3. My hand-drawn slope field for $p' = \frac{1}{2}p - 300$.

Moving towards numerical solutions to differential equations we ask ourselves 3 questions:

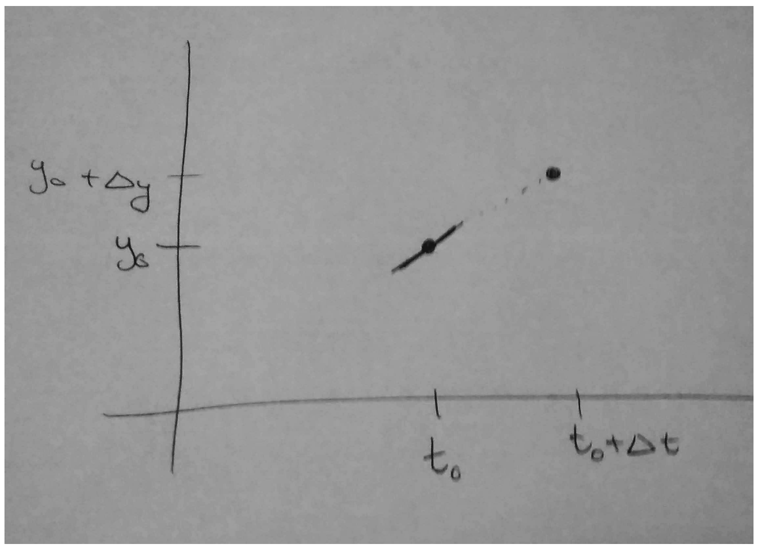
(1) Can we carry out the linking of tangent lines in a slope field picture in a straightforward and systematic way?
(2) If we can, is the resulting piece-wise linear function close to an actual solution?
(3) Can we measure the accuracy?

In this class we won't discuss question 3 much, but under certain reasonable conditions we can measure the accuracy. The answer to the first question is a resounding yes. Although the formulas are a little messy, they can be done.

Let's start by doing things graphically and locally. Consider the situation where $y(t_0) = y_0$ and y solves the differential equation

$$y' = f(t, y).$$

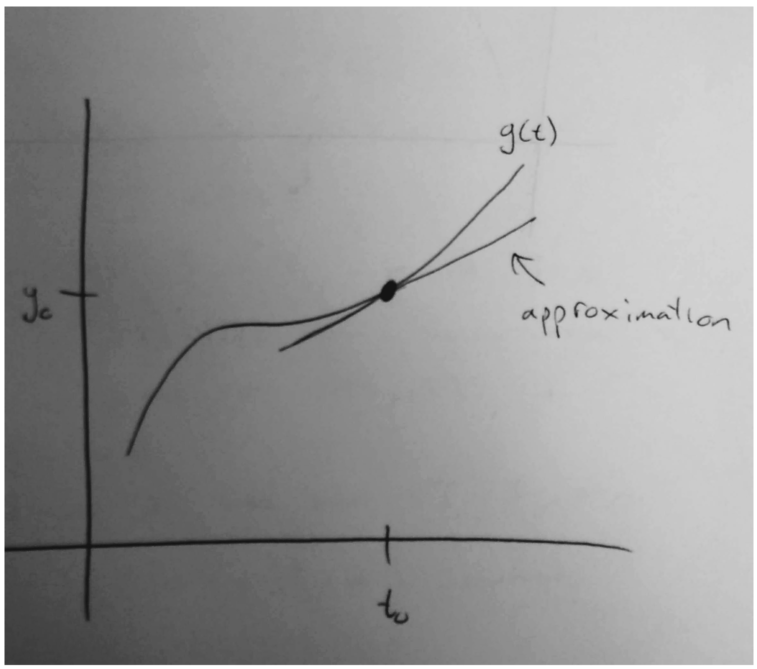
If Δt is a small number, then how do we estimate $y(t_0 + \Delta t)$? Let's look at the slope field (and ignore all the slope lines other than at the grid point (t_0, y_0)). The only line segment drawn below is has slope $f(t_0, y_0)$.



5. Lecture 5

Euler's Method: Often, explicitly solving the initial value problem $y' = f(t, y)$ and $y(t_0) = y_0$ is impossible, but we still want to approximate solutions. Last time we discussed slope fields, and now we will discuss a numerical method to solve this.

Let's start with a Math 124 example. That is, we suppose $y(t) = g(t)$ and $y(t_0) = y_0$. The tangent line that goes through the point (t_0, y_0) can be found explicitly:



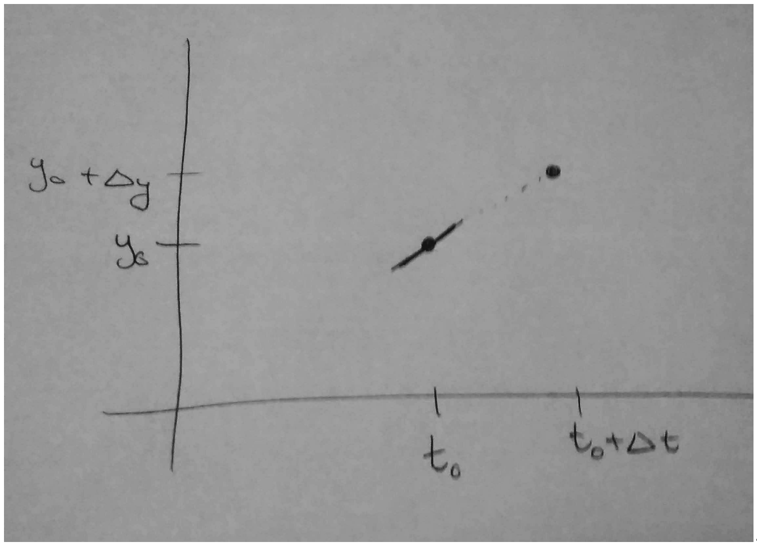
The approximating line has a slope of $g'(t_0)$ and goes through the point (t_0, y_0). Using the point-slope formula we get the tangent line is:

$$\text{tangent line is:} \qquad y_0 + g'(t_0) \cdot (t - t_0).$$

And this is a good approximation in some neighborhood of t_0, i.e. for $|h| \leq \Delta t$ we have:

$$y(t_0 + h) \approx y_0 + g'(t_0) \cdot (t_0 + h - t_0) = y_0 + g'(t_0) \cdot h.$$

We reprint the photo from the last lecture:



.

We guess that $y_0 + \Delta y$ should be approximately $y(t_0 + \Delta t)$, that is

$$y(t_0 + \Delta t) \approx y_0 + \Delta y.$$

Using the using the equation for the slope of a line, we can say that if

$$\frac{\Delta y}{\Delta t} = \frac{y_0 + \Delta y - y_0}{t_0 + \Delta t - t_0} = \text{slope of the line} = f(t_0, y_0).$$

Multiplying the outermost terms by Δt we get

$$\Delta y = f(t_0, y_0) \cdot \Delta t.$$

If $t = t_0 + \Delta t$ (for a small Δt) then we guess that the solution to the initial value problem:

$$\text{IVP:} \qquad \begin{cases} \frac{dy}{dt} &= f(t, y) \\ y(t_0) &= y_0 \end{cases},$$

is approximately

$$y(t) \approx y_0 + f(t_0, y_0) \cdot (t - t_0).$$

Let's work through an example together, before discussing the general method:

Example 5.1. Suppose that $y' = 3 + 2t$ and $y(1) = 0$. Estimate, using Euler's method, $y(2)$ using $\Delta t = 1, \frac{1}{2}$ or 0.2.

***Solution*:** Using the approximation $y(t) \approx y_0 + f(t_0, y_0) \cdot (t - t_0)$, and in particular $y(t_0 + \Delta t) \approx y_0 + f(t_0, y_0) \cdot \Delta t$. We will use this equation over and over.

$\Delta t = 1$:

$$y(2) \approx 0 + (3 + 2 \cdot 1) \cdot 1 = 5.$$

$\Delta t = \frac{1}{2}$.

$$\begin{aligned} y(1.5) &\approx 0 + (3 + 2 \cdot 1) \cdot 0.5 = 2.5 \\ y(2) &\approx 2.5 + (3 + 2 \cdot 1.5) \cdot 0.5 = 2.5 + 3 = 5.5 \end{aligned}$$

$\Delta t = 0.2$

$$\begin{aligned} y(1.2) &\approx 0 + (3 + 2 \cdot 1) \cdot 0.2 = 1 \\ y(1.4) &\approx 1 + (3 + 2 \cdot 1.2) \cdot 0.2 = 2.08 \\ y(1.6) &\approx 2.08 + (3 + 2 \cdot 1.4) \cdot 0.2 = 3.24 \\ y(1.8) &\approx 3.24 + (3 + 2 \cdot 1.6) \cdot 0.2 = 4.48 \\ y(2) &\approx 4.48 + (3 + 2 \cdot 1.8) \cdot 0.2 = 5.76 \end{aligned}$$

We can actually solve this differential equation:

$$y(t) = \int_1^t (3 + 2t)\, dt = 3t + t^2 - 4$$

and so $y(2) = 6$.
$\square$

We can actually create a piece-wise linear approximation to the actual solution. That is suppose we are given the following information:

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0, \qquad \Delta t \text{ is given.}$$

We want to find an approximation to the solution of the differential equation for $t_0 \leq t \leq t_N = t_0 + N\Delta t$. We use the above approximations over and over again.

Let $t_1 = t_0 + \Delta t$, $t_2 = t_0 + 2\Delta t$, $t_3 = t_0 + 3\Delta t$ and so on until $t_N = t_0 + N\Delta t$. Using the above computations we know that approximately:

$$y(t) \approx y(t_j) + f(t_j, y(t_j)) \cdot (t - t_j) \qquad \text{for } t_j \leq t \leq t_{j+1}.$$

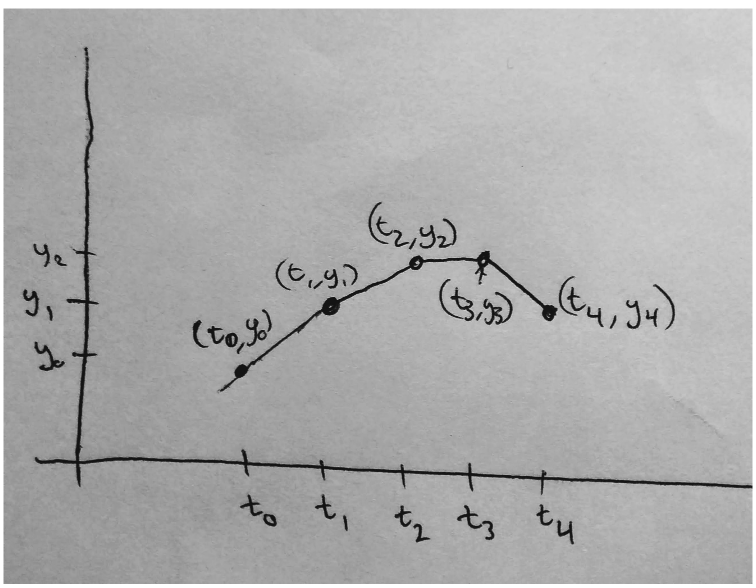
This gets quite ugly, so what we'll do is define

$$\begin{aligned} y_1 &= y_0 + f(t_0, y_0) \cdot \Delta t \\ y_2 &= y_1 + f(t_1, y_1) \cdot \Delta t \\ &\vdots \\ y_{j+1} &= y_j + f(t_j, y_j) \cdot \Delta t \\ &\vdots \\ y_N &= y_{N-1} + f(t_{N-1}, y_{N-1}) \cdot \Delta t. \end{aligned}$$

Then the approximate solution (in words) is

$$y(t) \approx \begin{pmatrix} \text{the } y\text{-value on the line} \\ \text{between } (t_j, y_j) \text{ and } (t_{j+1}, y_{j+1}) \\ \text{evalutated at } t \text{ when} \\ t \text{ is between } t_j \text{ and } t_{j+1} \end{pmatrix}.$$

Here's a (poorly) hand-drawn image:



Let's consider another initial value problem:

$$\text{IVP:} \quad \begin{cases} \dfrac{dy}{dt} &= f(t, y) = 3 - 2t - \frac{1}{2} y \\ y(0) &= 0 \end{cases}$$

Let's use Euler's method to approximate a solution. Since $y(0) = 0$ we get $t_0 = 0$ and $y_0 = 0$. The only thing else we have to specify is Δt, and let's set this up as $\Delta t = 0.1$ and $N = 10$. This is a lot of work to do by hand, but with the help of a calculator is not that bad if we are careful.

Let's start with compute (t_1, y_1) by hand. We know that $f(t_0, y_0) = 3$ and $\Delta t = .1$ so we get $(t_1, y_1) = (0.1, 0.3)$. For (t_2, y_2) we get

$$y_2 = y_1 + f(t_1, y_1) \cdot \Delta t = 0.3 + \left(3 - 2 \cdot 0.1 - \frac{1}{2} \cdot 0.3\right) \cdot 0.1 = 0.565$$

and so $(t_2, y_2) = (0.2, 0.565)$. If we compute the rest, we'll get the following table:

the index j	value of t_j	value of y_j
0	0	0
1	0.1	0.3
2	0.2	0.565
3	0.3	0.79675
4	0.4	0.9969125
5	0.5	1.16706688
6	0.6	1.30871353
7	0.7	1.42327785
8	0.8	1.51211396
9	0.9	1.57650826
10	1.0	1.61768285

As you can see the number get uglier as we go through the list.

On the next page, I include two images, one is the approximate solution when $\Delta t = 0.1$ and $N = 20$ (I change it for a larger picture) and $y(0) = 0$. As you'll see the images are fairly close to each other, and some of the details of the actual solution are apparent in the Euler's method solution.

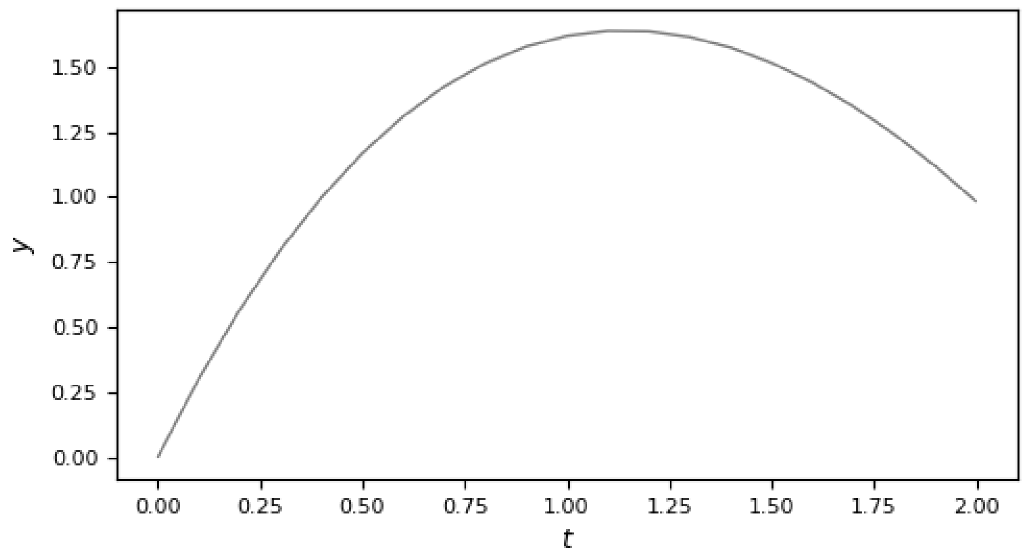


FIGURE 4. Euler's Method

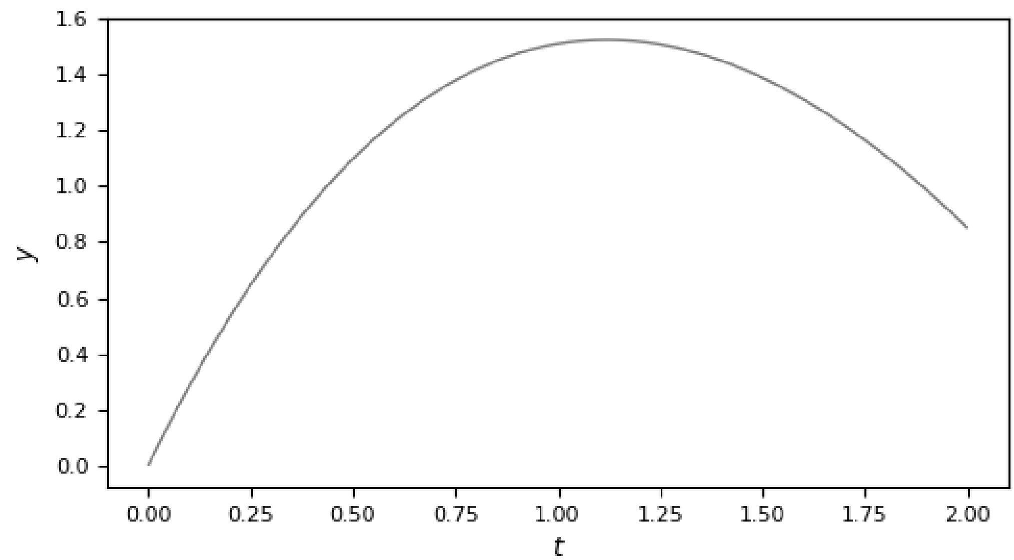


FIGURE 5. Actual solution

Here is the Python code for the above simulations:

```
import numpy as np
import pandas as pd

def exportToCSV(dataSet, filename):
    my_df = pd.DataFrame(dataSet)
    my_df.to_csv(filename+".csv", index=False, header=False )

def f(t,y):
    return 3-2*t-float(y)/2

def solution(start,end,dt):
    size = (float(end)-float(start))/dt
    size = int(size)
    t = np.linspace(0,end,size)
    y = np.zeros(np.size(t))
    for j in range(0,size):
        y[j] = 14-4*t[j]-14*np.exp(-float(t[j])/2)
    return joinXandY(y,t)

def joinXandY(X,t):
    l = np.size(X)
    data = np.array([(t[j],X[j]) for j in range(0,l)])
    return data

def EulerMethod(dt,t_0,y_0,N):
    t = np.linspace(t_0,t_0+N*dt,int(N+1))
    y = np.zeros(np.size(t))
    y[0] = y_0
    for j in range(1,N+1):
        y[j] = y[j-1]+ f(t[j-1],y[j-1])*dt
    return joinXandY(y,t)

X = EulerMethod(.1,0,0,20)
Y = solution(0,2,.01)
exportToCSV(X,"EulerMethod")
exportToCSV(Y,"Solution")
```

6. Lecture 6

There is one last note I want to make on Euler's method. While using Euler's method you must keep track of all the data you have, and do that in a systematic way. When I do Euler's method by hand I generally keep a table of information, sort of like the one included a few pages ago. Typically, I include more information than that. For example I would include a column for finding $f(t, y)$ which would allow me to lower the likelihood of making an algebraic mistake. For example you can have a table with the following columns:

index: j	t_j	y_j	$f(t_j, y_j)$	Δy	the next y: y_{j+1}

This is not the only way of organizing the information, you can omit columns as you see fit or even add additional columns.

We are now moving on to **linear first order differential equations**. Let's break down what these equations "look like." The first order part of the differential equation means that we deal with equations involving y' and the linear part means that we can write it in the form:

$$y' + p(t)y = g(t),$$

where p and g are functions only of the time variable.

We have already encountered these when p and g are constants. This was done in Lecture 2. The way we solve these equations is by using something called an **integrating factor**. We'll use an integrating factor to solve a specific case.

Example 6.1. Solve the differential equation:

$$\frac{dy}{dt} + \frac{1}{2}y = \frac{1}{2}e^{t/3}.$$

***Solution*:** To use the method of integrating factors we multiply the entire equation by a function $\mu(t)$ to give us:

$$\mu(t)y' + \frac{1}{2}\mu(t)y = \frac{1}{2}\mu(t)e^{t/3}.$$

This may make the equation look harder, but if we can choose $\mu(t)$ so that the left-hand side is $\frac{d}{dt}[\mu(t)y]$ then we would be able to integrate both sides to get the solution. Elaborating on this a little, we suppose that we can find a μ

$$\frac{d}{dt}[\mu(t)y] = \mu(t)y' + \frac{1}{2}\mu(t)y = \frac{1}{2}\mu(t)e^{t/3}.$$

IF a μ exists (and that is a big if) then we can multiply the outermost terms by dt and integrate to get:

$$\mu(t)y = \int d(\mu(t)y) = \int \frac{1}{2}\mu(t)e^{t/3}\,dt,$$

which will give us:

$$y = \frac{1}{\mu(t)} \int \frac{1}{2}\mu(t)e^{t/3}.$$

Let's now compute the derivative of $\mu(t)y$. By the product rule we get:

$$\frac{d}{dt}\left[\mu(t)y\right] = \mu(t)y' + \mu'(t)y.$$

We want this to be equal to $\mu(t)y' + \frac{1}{2}\mu(t)y$. The y' terms already match up, so we just want to solve

$$\frac{1}{2}\mu(t)y = \mu'(t)y \qquad \text{which is} \qquad \mu'(t) = \frac{1}{2}\mu(t).$$

We know how to solve the differential equation:

$$\frac{d\mu}{dt} = \mu' = \frac{1}{2}\mu,$$

by separating variables. So

$$\begin{aligned}
\frac{d\mu}{dt} &= \frac{1}{2}\mu \\
\frac{d\mu}{\mu} &= \frac{1}{2}\,dt \\
\ln|\mu(t)| = \int \frac{d\mu}{\mu} &= \int \frac{1}{2}\,dt = \frac{1}{2}t + C \\
|\mu(t)| &= e^C e^{t/2} \\
\mu(t) &= \pm e^C e^{t/2} = Ae^{t/2}.
\end{aligned}$$

We don't need a general solution to this differential equation, we just need 1 particular solution so we can choose $A = 1$.

Now we get to:

$$\begin{aligned}
e^{t/2}y &= \int \frac{1}{2}e^{t/2}e^{t/3}\,dt = \int \frac{1}{2}e^{5t/6}\,dt \\
&= \frac{3}{5}e^{5t/6} + c.
\end{aligned}$$

Dividing everything by $e^{t/2}$ and using properties of exponents yields:

$$y = \frac{3}{5}e^{t/3} + ce^{-t/2}.$$

□

We can actually extend that method to equations of the form:

$$y' + ay = g(t).$$

We multiply everything by $\mu = \mu(t)$ chosen so that:

$$\frac{d}{dt}[\mu y] = \mu y' + a\mu y = \mu g(t),$$

but actually differentiating μy gives

$$\frac{d}{dt}[\mu y] = \mu y' + \mu' y,$$

and so $\mu' = a\mu$, so we can take $\mu(t) = e^{at}$.

That means

$$\frac{d}{dt}(e^{at}y) = e^{at}g(t) \qquad \text{which means} \qquad e^{at}y = \int e^{at}g(t)\,dt,$$

where that integral has a constant of integration that will appear. We can get arround that by doing:

$$e^{at}y = \int_{t_0}^{t} e^{as}g(s)\,ds + c.$$

Question: What is c?

Therefore:

$$y = e^{-at}\int_{t_0}^{t} e^{as}g(s)\,ds + ce^{-at}.$$

7. Lecture 7

Let's review integrating factors. Consider the differential equations:

$$t^3y' + 3t^2y = \cos(t)$$
$$e^{-t}v' - e^{-t}v = 4t^2 + e^{2t}.$$

The left hand sides are the derivatives of the following:

$$(t^3y) \qquad \text{and} \qquad e^{-t}v.$$

Thus we get:

$$t^3y = \int \cos(t)\,dt = \sin(t)+c \qquad \text{and} \qquad e^{-t}v = \int 4t^2+e^{2t}\,dt = \frac{4}{3}t^3+\frac{1}{2}e^{2t}+c.$$

From there we can easily solve for the general solutions.

Let's return to the general problem:

$$y' + p(t)y = g(t).$$

We carry through everything the same way.

(1) Multiply by $\mu = \mu(t)$:

$$\mu y' + \mu p(t)y = \mu g(t).$$

(2) Differentiate (μy):

$$\frac{d}{dt}[\mu y] = \mu y' + \mu' y$$

and match it with

$$\mu y' + \mu p(t)y.$$

(3) Isolate μ' by looking that y "coefficients":

$$\mu' = \mu p(t).$$

(4) Solve for a particular solution of μ:

$$\begin{aligned}\frac{d\mu}{dt} &= \mu p(t)\\ \frac{d\mu}{\mu} &= p(t)\,dt\\ \ln|\mu| &= \int p(t)\,dt\\ |\mu| &= \exp\left(\int p(t)\,dt\right)\\ \mu &= \pm\exp\left(\int p(t)\,dt\right),\end{aligned}$$

where $\exp(x) = e^x$. We can ignore the $\pm$ term to get

$$\mu(t) = \exp\left(\int p(t)\,dt\right).$$

(5) Since we now have a μ that works we have to solve:

$$\frac{d}{dt}[\mu y] = \mu g(t),$$

which is done by

$$\mu y = \int_{t_0}^{t} \mu(s)g(s)\,ds + c.$$

(6) Solve for y to get

$$y = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s)g(s)\,ds + c\right],$$

which is the simplified version of

$$y(t) = \exp\left(-\int p(t)\,dt\right)\left[\int_{t_0}^{t} \exp\left(-\int p(s)\,ds\right) g(s)\,ds + c\right].$$

Example 7.1. Solve the initial value problem:

$$\text{IVP:} \qquad \begin{cases} y' &= 2ty + t \\ y(0) &= 1 \end{cases}.$$

***Solution*:** We have:

$$y' - 2ty = t.$$

That means $p(t) = -2t$ and so the integrating factor must be:

$$\mu(t) = \exp\left(\int p(t)\right) = \exp\left(-t^2\right),$$

where we can ignore the constant of integration inside exp.

We now have:

$$(\mu y)' = t\mu(t) = te^{-t^2}.$$

Multiplying by dt on both sides and integrating gives

$$e^{-t^2} y = \int te^{-t^2}\,dt,$$

where there is a constant of integration hidden in the right-hand side.

We can write one solution to the RHS above as:

$$\int_0^t se^{-s^2}\,ds + c.$$

Using $u = -s^2$, $du = -2s\,ds$ makes the integral equal to

$$\int_0^{-t^2} \frac{-1}{2} e^u\,du = \frac{-1}{2} e^u \Big|_{u=0}^{u=-t^2} = \frac{1}{2}\left(1 - e^{-t^2}\right).$$

This gives us:

$$e^{-t^2} y = \frac{1}{2}\left(1 - e^{-t^2}\right) + c = c_2 - \frac{1}{2} e^{-t^2}.$$

Using $y(0) = 1$ we get:

$$1 = e^0 y(0) = c_2 - \frac{1}{2} e^0 = c_2 - \frac{1}{2}$$

or, $c_2 = \frac{3}{2}$.

Thus the solution to the initial value problem is:

$$y(t) = \frac{3}{2} e^{t^2} - \frac{1}{2}.$$

□

Example 7.2. Solve the differential equation:

$$y' + y = 5\sin(t).$$

Solution: We have $p(t) = 1$ and so $\mu(t) = e^t$. Thus we must integrate (technically $5\times$ this):

$$\int e^t \sin(t)\, dt.$$

We use integration by parts with $u = e^t$ and $dv = \sin(t)\, dt$. This gives:

$$\int e^t \sin(t)\, dt = -\cos(t)e^t + \int \cos(t)e^t\, dt.$$

We again use integration by parts to get:

$$\int \cos(t)e^t\, dt = \sin(t)e^t - \int \sin(t)e^t\, dt.$$

Therefore:

$$\int e^t \sin(t)\, dt = \sin(t)e^t - \cos(t)e^t - \int \sin(t)e^t\, dt,$$

$$\int \sin(t)e^t\, dt = \frac{e^t}{2}\left(\sin(t) - \cos(t)\right) + C,$$

where we add the constant of integration because we are doing an indefinite integral on the left-side so we know there must be a "$+C$" term on the right-hand side. Therefore:

$$e^t y = \frac{5}{2}(\sin(t) - \cos(t))e^t + c$$

which implies

$$y = \frac{5}{2}(\sin(t) - \cos(t)) + ce^{-t}.$$

□

Example 7.3. Find the integrating factor for the following ODE's

(1) $y' - y = 2te^t$
(2) $ty' - 2y = t$ for $t > 0$.
(3) $ty' + (t+1)y = 7$ for $t > 0$.
(4) $y' - 6y = -t^2 y' + 4e^t$.

8. Lecture 8

Here is the problem we ended class with during Lecture 7:

5. A small island in the Pacific is having a problem with its invasive rat population. The residents of the island notice that the rat population is growing at a rate proportional to its own size, increasing in size by a factor 2 every 8 months. To counter this, the residents bring in a shipment of cats, who upon arrival catch and kill rats at an initial rate of 625 a month. However, the cats grow more skilled in their rat-catching efforts as time goes on, and as such the number of rats they catch increases by 25 every month. The cats arrive at the beginning of the year, when there are 3800 rats on the island.
Set up an inital value problem for the population $P(t)$ of rats at time t in months. (DO NOT SOLVE).

Let's work through a solution to part (a) together. We can write in words that:

$$\frac{dP}{dt} = \text{reproduction rate} - \text{killing rate}.$$

Let's isolate just the reproduction rate first. That is, let $S(t)$ be the number of rats on the island at time t IF THERE WERE NO CATS on the island. Then we would have:

$$\frac{dS}{dt} = \text{reproduction rate}.$$

The problem states that the rats reproduce at a rate proportional to its own size and so we can write:

$$\frac{dS}{dt} = rS \qquad \text{or, after solving} \qquad S(t) = Ae^{rt}.$$

The problem also states that every 8 months the rat population will double (if there is no killing by the cats). In an equation, this means:

$$\begin{aligned} S(t+8) &= 2S(t) \\ Ae^{r(t+8)} &= 2Ae^{rt} \\ \frac{Ae^{r(t+8)}}{Ae^{rt}} &= 2 \\ e^{8r} &= 2. \end{aligned}$$

Solving for r gives $r = \frac{1}{8}\ln(2)$.

The problem states that the cats initially catch 625 rats per month but then increase that number by 25 rats per month. That means by month t, the cats kill $625 + 25t$ rats.

Finally, there are 3,800 rats at the start of the year.

We thus arrive at the initial value problem:

$$\frac{dP}{dt} = \frac{\ln 2}{8}P - (625 + 25t), \qquad P(0) = 3800.$$

We now turn to the topic of §2.3, Modeling with First Order Equations.

Differential equations are useful. For mathematicians differential equations just always seem to pop up in areas of research.

For non-mathematicians differential equations are useful because it is easier to change parameters in some equation and solve that equation than it is to rerun an experiment with different conditions. This is both cheaper and more time-efficient. This is not perfect. There is a downside with this process. Physical systems are not always perfect, these mathematical models are just an approximate description of the actual reality. When you mix chemicals the reactions don't happen instantaneously, some happen really fast but there is a small delay.

We'll walk through some examples together in class, and also I'll type up more than what we cover in class for extra review.

Example 8.1. Louis wants to buy a home. He can afford to spend no more than \$1500 per month on a mortgage payment. Unfortunately, the only banks that will loan to him have a weird policy. They'll give him a loan with an interest rate of 6% compounded continuously, but they take money out of his bank account at a continuous rate as well. For simplicity, we assume that there are 30 days per month. What is the maximum loan he can take out for a 30-year mortgage. (Alternatively, we can think of Louis as borrowing money from a bank that uses the Ethopian calendar and completely shuts down for during the intercalary month of Paguemain).

***Solution*:** Let $M(t)$ denote the amount of money that Louis owes t years into his mortgage. We want to figure out what $M(0)$ can be if $M(30) = 0$.

Let's write a differential equation in words:

$$\frac{dM}{dt} = \text{interest} - \text{payments}.$$

Since interest accrues continuously, the interest term looks like interest $=$ rM where r is the interest rate. Moreover, each year Louis pays exactly

$12 \times 1500 = 18,000$ dollars. Thus the payment term is payment $= 18000$. This makes the differential equation

$$\frac{dM}{dt} = .06M - 18000.$$

This is a **separable differential equation** which we know how to solve. We go through the steps:

$$\begin{aligned} \frac{dM}{dt} &= .06M - 18000 \\ \frac{dM}{.06M - 18000} &= dt \\ \frac{1}{0.06}\ln|0.06M - 18000| = \int \frac{dM}{.06M - 18000} &= \int dt = t + C \\ \ln|.06M - 18000| &= .06t + C \\ |.06M - 18000| &= e^{.06t+C} = e^C e^{.06t} \\ .06M &= 18000 \pm e^C e^{.06t} \\ M(t) &= 300,000 + Ae^{.06t} \end{aligned}$$

where

$$A = \frac{\pm e^C}{0.06}.$$

We now must find A using the initial condition which is (in words) the mortgage is paid off in 30 years. So

$$0 = M(30) = 300,000 + Ae^{0.06 \cdot 30} = 300,000 + Ae^{1.8}.$$

Therefore,

$$A = -\frac{300,000}{e^{1.8}} \approx -49,589.67\,.$$

From here we get

$$M(t) = 300,000 - 49,589.67e^{0.06t}.$$

So

$$M(0) = 300,000 - 49,589.67 = 250,410.33$$

is the maximum loan he can take from the bank. □

Example 8.2. Important archeological research is based on radiocarbon dating, particularly of carbon-14. This can be used to determine the age of some plant remains up to 50,000 years after the vegetation dies. The half-life of an isotope is defined as the amount of time it takes for half of large collection of that isotope to decay into something else. The half-life of carbon-14 is approximately 5,730 years. An substance which undergoes radioactive decay, decays at a rate proportional to the amount of the substance. That is $Q' = -rQ$ where r is the rate of decay which depends on the type of particle and Q is the quantity (i.e. mass) of the substance. Find the r value for carbon-14.

***Solution*:** Let Q be the quantity of carbon-14. We know that

$$Q(5730) = \frac{1}{2}Q(0),$$

since every 5730 years half of a sample of carbon-14 will decay. We can show that

$$Q(t) = Q_0 e^{-rt}$$

where $Q(0) = Q_0$. Therefore, we get:

$$Q_0 e^{-r\cdot 5730} = Q(5730) = \frac{1}{2}Q_0.$$

Which means

$$e^{-r\cdot 5730} = \frac{1}{2}.$$

Taking ln's and dividing by -5730 gives:

$$r = \frac{-\ln(1/2)}{5730} = \frac{\ln 2}{5730} \approx 1.21 \times 10^{-4}.$$

□

Example 8.3. Suppose a population of bacteria in a culture grows at a rate proportional to the number of bacteria present at time t. After 3 hours it is observed that there are 400 bacteria. After 10 hours 2000 are preset. What is the initial number of bacteria?

***Solution*:** Let $B(t)$ be the number of bacteria present at time t. It says that B increases at a rate proportional to the number of bacteria so

$$B' = rB \qquad \text{where } r \text{ is a proportionality constant}$$

You can solve ODE by seperation of variables, and we can see it gives

$$B = Ae^{rt}.$$

Here there are two unknowns A and r. We do have two relations that we know $B(3) = 400$ and $B(10) = 2000$. Using these we get the equations:

$$Ae^{3r} = 400 \qquad \text{and} Ae^{10r} = 2000.$$

We can divide the equations to get

$$e^{7r} = \frac{Ae^{10r}}{Ae^{3r}} = \frac{2000}{400} = 5$$

we can solve for r to get

$$r = \frac{1}{7}\ln 5 \approx 0.23$$

Now using $B(3)$ (or $B(10)$) we can find

$$Ae^{3\cdot(0.23)} = 400 \qquad \text{or} \qquad A = \frac{400}{e^{0.23}} \approx 201.$$

□

Example 8.4. A skydiver weighing 180 lbs (including equipment) jumps out of a plane from a height of 5,000 ft and opens the parachute after 10 seconds of free fall. Assume the magnitude of the force due to air resistance is $\frac{3}{4}|v|$ without a parachute, but is $12|v|$ when the parachute is open. Measure v is ft per second. Use $g = 32$ ft/s.

(1) Find the speed of the skydiver when the parachute opens.
(2) Find the distance traveled before the parachute opens.
(3) What is the terminal velocity after opening the parachute?

***Solution*:**

(1) Let $y(t)$ be the altitude in feet at time t seconds. Then $v = y'$ is the velocity and $a = v'$ is the acceleration.

The forces are

$$F = -mg - \frac{3}{4}v$$

The mg terms is negative because the positive direction for y is upwards and the force of gravity points downwards. The $\frac{3}{4}v$ term has a minus sign in front because air-resistance opposes motion, that is the force of drag points in the opposite direction of velocity.

The term mg is precisely the weight of the skydiver and equipment $mg = 180$ and $g = 32$ ft/s and so the mass is

$$m = 180/32 = \frac{45}{8}.$$

Using Newton's second law we get:

$$mv' = -mg - \frac{3}{4}v \qquad \text{or} \qquad v' = -32 - \frac{2}{15}v.$$

This is both a separable ODE and linear first order ODE so we can solve it either way. Let's do integrating factors:

$$v' + \frac{2}{15}v = -32.$$

The integrating factor is

$$\mu(t) = \exp\left(\int p(t)\, dt\right) = e^{\frac{215t}{\cdot}}$$

Therefore:

$$\frac{d}{dt}\left(e^{2t/15}v\right) = e^{2t/15}\frac{dv}{dt} + e^{2t/15}v = -32e^{2t/15}$$

Thus

$$e^{2t/15}v = \int -32e^{2t/15}\,dt = -240e^{2t/15} + C.$$

That means

$$v = -240 + Ce^{-2t/15}.$$

Since the skydiver falls, $v(0) = 0$ and this tells us that $C = 240$. When the parachute opens at $t = 10$ seconds we have

$$v(t) = -240 + 240e^{-2\cdot 10/15} \approx -176.7\text{ft/s}.$$

Since speed is the absolute value of velocity we have

$$\text{the speed when the parachute opens is } 176.7\frac{\text{ft}}{\text{s}}.$$

(2) Distance traveled is $\int |v(s)|\,ds$. Thus

$$\begin{aligned}
\text{distance} &= \int_0^{10} \left|240e^{-2t/15} - 240\right|\,dt \\
&= \int_0^{10} 240 - 240e^{-2t/15}\,dt \\
&= 240t + 1800e^{-2t/15}\Big|_{t=0}^{t=10} \\
&= \left(2400 + 1800e^{-4/3}\right) - (0 + 1800) \\
&\approx 600 + 474.5 = 1074.5
\end{aligned}$$

(3) To simplify the equations, let's reset time at $t = 0$ when the parachute opens. Changing the differential equation appearing in part 1, gives:

$$\frac{45}{8}v' = -180 - 12v,$$

with the intial condition $v_0 = -176.7$. Rearranging the differential equation it becomes

$$v' + \frac{32}{15}v = -32,$$

which means the integrating factor is $\mu(t) = e^{32t/15}$.

Therefore we can get

$$e^{32t/15}v = \int -32e^{32t/15}\,dt = -15e^{32t/15} + C.$$

Meaning

$$v(t) = -15 + Ce^{-32t/15}.$$

Using $v(0) = -176.7$ gives

$$-176.7 = -15 + Ce^{0} \qquad \text{and} \qquad C = -161.7.$$

Therefore

$$v(t) = -15 - 161.7e^{-32t/15}.$$

There terminal velocity is

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty}\left(-15 - 161.7e^{-32t/15}\right) = -15.$$

□

9. Lecture 9

An **autonomous** differential equation is a differential equation of the form:

$$y' = f(y).$$

Since y' does NOT depend on t, the slope fields are translation invariant, i.e. the slope only changes as y changes. Autonomous differential equations often show up in population dynamics problems.

If $f(c) = 0$ then $y = c$ is a solution to the differential equation since $\frac{dy}{dt} = 0$ and $f(c) = 0$ as well. If $f(c) = 0$, then $y = c$ is an **equilibrium solution**. The behavior of f nearby, determines something called the **stability** of the solution.

We can study the behavior of solutions to the differential equation $y' = f(y)$ by looking at the behavior of f. By looking at where $f(y)$ is positive, negative or zero we can study whether or not y is increasing or decreasing.

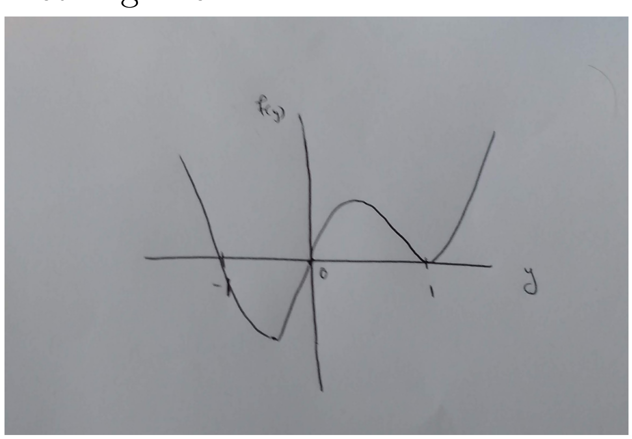
Example 9.1. Consider the autonomous differential equation

$$y' = 3y(1 + y)(1 - y)^2.$$

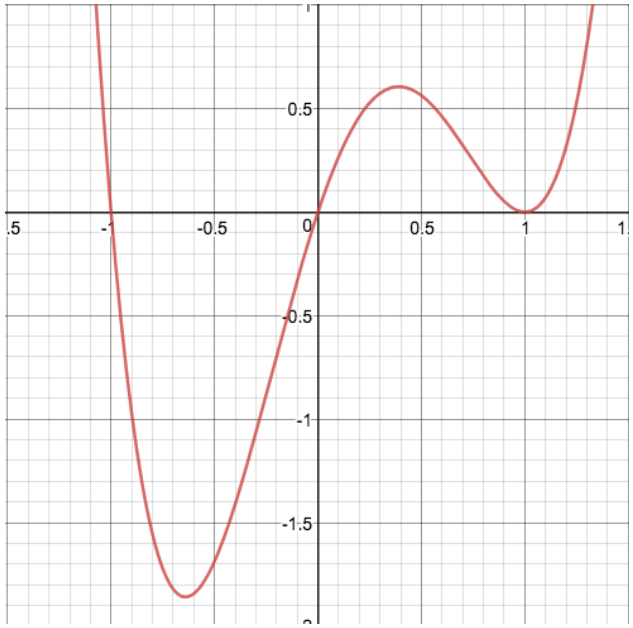
If $y(0) = y_0$, then for what values of y_0 will the solution be increasing, decreasing or constant?

***Solution*:** This example is $f(y) = 3y(1+y)(1-y)^2$. Instead of solving this differential equation explicitly, we'll just look at the function f. We know that $f(0) = f(-1) = f(1) = 0$ and so we just have to find out when f is positive and negative. By the intermediate value theorem, we know that if $f(a) > 0$ and $f(b) < 0$ then for some c between a and b, $f(c) = 0$.

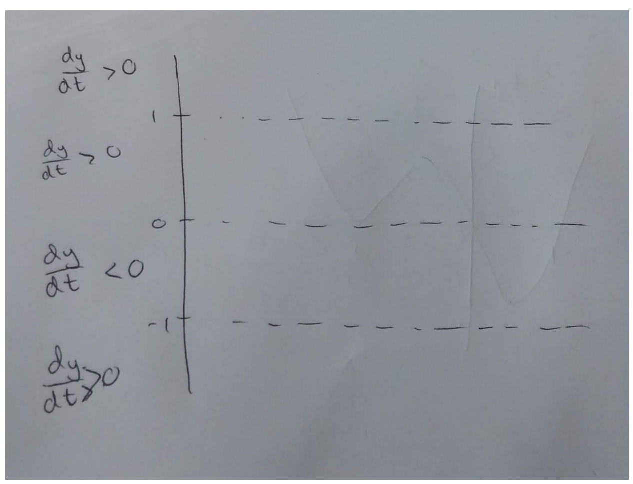
We can see that $f(.5) > 0$, $f(-.5) < 0$, $f(-2) > 0$ and $f(2) > 0$ so the graph must look something like:



As you can tell, we don't have any scale on the vertical axis. We don't actually need that to get information about solutions. Specifically, we don't need to know what the actual values of f are, we just need to know the sign. For those curious, here is a graph of the function f:



Once we know the signs of f we can draw something called a **phase line diagram** which looks like this:



Solutions cannot cross these phase lines. That means if y_0 is between 0 and 1 then $0 < y(t) < 1$ for all t.

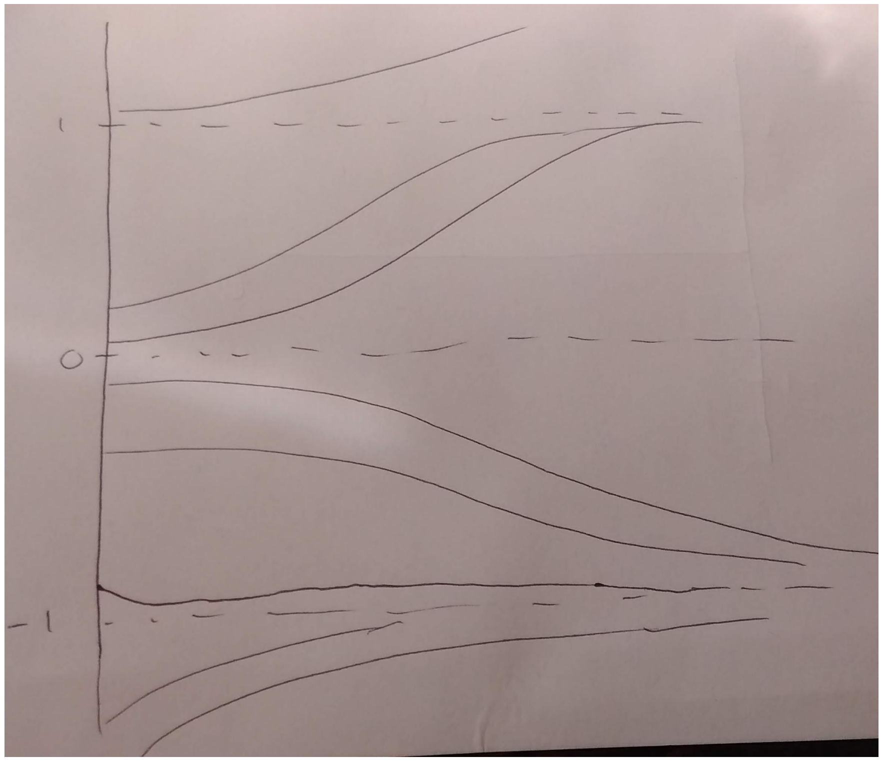
Let's analyze what's happening in the phase line diagram. When $y_0 > 1$ then y' is always positive, so y increases with t, in fact $y(t) \to \infty$ as $t \to \infty$.

If $0 < y_0 < 1$ then $y' > 0$ so the solution increases. Since a solution cannot cross the phase lines, the solution $y(t)$ converges to 1 as $t \to \infty$.

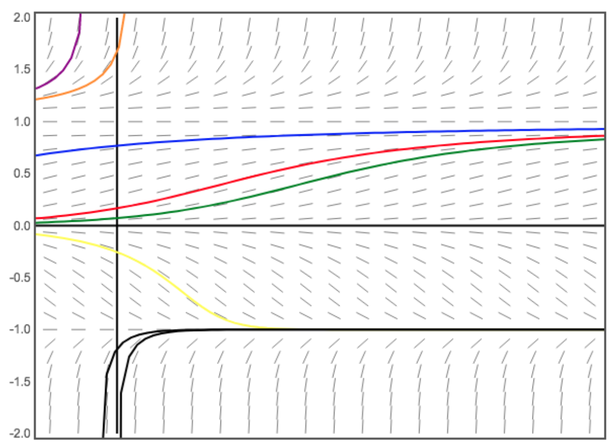
If $-1 < y_0 < 0$ then $y' < 0$ and so the solution decreases. Since a solution cannot cross the phase lines, $y(t) \to -1$ as $t \to \infty$.

If $y_0 < -1$ then $y' > 0$ and so $y(t)$ increases towards -1 as $t \to \infty$.

That means solutions look something like the following:



Here is a graph of the actual direction field along with some of the **integral curves** (a picture of a solution to the differential equation) drawn:



Since solutions which start near -1 will converge to -1 as $t \to \infty$ we call $y = -1$ a **stable solution**.

Since solutions which start near 0 will move away from 0 we call 0 an **unstable solution**.

Since some solutions which start near 1, will converge to 1 while some will go away from 1 we call $y = 1$ a **semi-stable solution**. $\square$

Let's jump back to studying general autonomous differential equations. Suppose that

$$y' = f(y),$$

we want to find out which equilibrium solutions are stable, and which are unstable and semi-stable. The first thing we do is find the roots of f, that is the c's such that $f(c) = 0$. The next thing we can check is whether or not $f'(c)$ is positive, negative or 0.

If $f'(c) > 0$ then the solution is unstable, if $f'(c) < 0$ the solution is stable. If $f'(c) = 0$ then we cannot tell whether the solution is stable, unstable or semi-stable (in fact there are examples where it could be either).

10. Lecture 10

In this lecture we'll cover **constant coefficient second order differential equations**. These are equations of the form

$$ay'' + by' + cy = 0.$$

Let's start with a simple example. Suppose

$$y'' - y = 0.$$

This means that the second derivative of y with respect to t is y itself. You probably know two functions that satisfy this differential equation. For example, $y(t) = 2e^t$ and $y(t) = 5e^{-t}$ both satisfy the above differential equation. In fact, we can write the general formula for a solution as

$$y(t) = Ae^t + Be^{-t},$$

and since there are two unknowns, we need to know to two initial values.

Example 10.1. Solve the initial value problem

$$\begin{cases} y'' - y &= 0 \\ y(0) &= 2 \\ y'(0) &= -1 \end{cases}.$$

***Solution*:** The general solution is of the form $y(t) = Ae^t + Be^{-t}$. By differentiating, we get $y'(t) = Ae^t - Be^{-t}$. Evaluating these two equations at 0 gives two different equations

$$2 = y(0) = Ae^0 + Be^0 = A + B$$
$$-1 = y'(0) = Ae^0 - Be^0 = A - B$$

We can solve these equations by, for example, adding the two equations together to get

$$1 = 2A \qquad \text{or} \qquad A = \frac{1}{2}.$$

This then allows us to find

$$B + \frac{1}{2} = 2 \qquad \text{so} \qquad B = \frac{3}{2}.$$

□

Let's move solving the general constant coefficient equation:

$$ay'' + by' + cy = 0.$$

We suspect that e^{rt} is a solution for some number r. Let's compute what $ay'' + by' + cy$ is when $y = e^{rt}$.

Well

$$(e^{rt})' = re^{rt} \qquad \text{and} \qquad \left(e^{rt}\right)'' = r^2 e^{rt}.$$

Thus

$$ay'' + by' + c = ar^2 e^{rt} + bre^{rt} + ce^{rt} = \left(ar^2 + br + c\right) e^{rt}.$$

Therefore, if we want $y = e^{rt}$ to solve the differential equation $ay'' + by' + cy = 0$, then we need

$$0 = (ar^2 + br + c)e^{rt}.$$

Since e^{rt} is never zero, we must have

$$ar^2 + br + c = 0.$$

Roughly speaking, the equation differential equation

$$ay'' + by' + cy = 0$$

boils down to finding the roots of the polynomial

$$ar^2 + br + c = 0.$$

We call that above polynomial the **characteristic equation**.

Assume that the roots of the characteristic equation $ar^2+br+c = 0$ has two **distinct** roots $r_1 \neq r_2$. Then the general solution of the differential equation

$$ay'' + by' + cy = 0$$

is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

The important thing is $r_1 \neq r_2$. If $r_1 = r_2$ is NOT the general solution, we'll cover this more later this week.

Example 10.2. Find the general solution to the differential equation

$$y'' + 5y' + 6y = 0.$$

And then solve the initial value problem

$$\begin{cases} y'' + 5y' + 6y &= 0 \\ y(0) &= 2 \\ y'(0) &= 3 \end{cases}.$$

Solution: The characteristic equation is

$$r^2 + 5r + 6 = (r + 2)(r + 3) = 0.$$

Thus

$$y(t) = Ae^{-2t} + Be^{-3t}$$

solves the differential equation.

Using the initial values

$$2 = y(0) = A + B$$
$$3 = y'(0) = -2A - 3B.$$

This can be solved explicitly for

$$A = 9 \qquad \text{and} \qquad B = -7.$$

Therefore the solution to the solution to the initial value problem is

$$y(t) = 9e^{-2t} - 7e^{-3t}.$$

□

Example 10.3. Find the solution to the initial value problem

$$\begin{cases} 2y'' + 2y' - 40y &= 0 \\ y(0) &= 2 \\ y'(0) &= -1 \end{cases}$$

Solution: The characteristic equation is

$$2r^2 + 2r - 40 = 2(r^2 + r - 20) = 2(r - 4)(r + 5) = 0.$$

This has two distinct roots, $r = 4$ and $r = -5$. That means the general solution is

$$y(t) = Ae^{4t} + Be^{-5t}.$$

Using the initial values

$$2 = y(0) = A + B$$
$$-1 = y'(0) = 4A - 5B.$$

This implies that $A = B = 1$, and so the solution to the initial value problem

$$y(t) = e^{4t} + e^{-5t}.$$

□

11. Lecture 11

Example 11.1. Setting up a spring-mass system.

Consider a mass m attached to a spring. The springs natural length is ℓ and when the mass is attached the length is $\ell + L$. Let $u(t)$ be vertical position of the mass m at time t.

What are the forces acting on m? Write a differential equation for u.

***Solution*:** Let $u(t)$ denote the vertical position of the mass at time t where u increases as we move upwards in the picture. We say that $u = 0$ when spring is at its natural length. Then $u'(t)$ is the velocity at time t and $u''(t)$ is the acceleration at time t.

Hooke's Law tells us that the force acting on a(n ideal) spring is proportional to the distance from the spring's natural length. That tells us that

$$F_{\text{spring}}(t) = -ku(t)$$

where $k \geq 0$ is some constant called the **spring constant**, which depends on the spring. The minus sign is there to tell you that when the mass m is below the natural length (i.e. $u(t) < 0$) and so we want the force to point upwards (the opposite direction of u).

There is also a damping force, which is proportional to the velocity at which the object is moving. When the speed is large (velocity is large in absolute value) the damping force acts to slow down the speed of the object. That means, the damping force looks like:

$$F_{\text{damping}}(t) = -\gamma u'(t).$$

The only other "natural" force acting on the object m is gravity, which is easier to write as

$$F_{\text{gravity}}(t) = -mg$$

and here the minus sign means that gravity is pointing downwards (the opposite direction of up).

That takes care of all the forces that would be found in nature, but we can also add a force to this system. This is called an external force, and is of the form

$$F_{\text{external}}(t) = F(t)$$

Since forces add, we get

$$F_{\text{total}} = F_{\text{spring}} + F_{\text{damping}} + F_{\text{gravity}} + F_{\text{external}}.$$

This gives a differential equation

$$mu'' = -ku - \gamma u' - mg + F(t)$$

or

$$mu'' + \gamma u' + ku = F(t) - mg.$$

□

Remark 11.1. THIS IS IMPORTANT: The right-hand side is based on the definition of where 0 was. If we changed where height $u = 0$ is, then the term on the right-hand side will (very very likely) change.

Back to second order linear constant coefficient equations. We work with the equation:

$$ay'' + by' + cy = 0.$$

The solution to this is based on the solutions to the polynomial equation

$$ar^2 + br + c = 0.$$

The quadratic equation tells us

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We now have the following break-downs:

(1) $b^2 - 4ac > 0$ then there are two distinct real roots and we already know how to solve this differential equation.
(2) $b^2 - 4ac = 0$ then there is a repeated root, and we will learn how to solve the differential equation sometime soon.
(3) $b^2 - 4ac < 0$ then there is are two distinct roots, but they are complex numbers (that is involve imaginary numbers). The formulas still work in this case, but we have to interpret what the exponential of an imaginary number is.

Let's deal first with the complex roots situation first. But in order to do this, I'll first need to state one important theorem

Theorem 11.1. *Suppose that f is a function of $(n+1)$-variables, that is*

$$f(x_0, \ldots, x_n).$$

Consider the initial value problem

$$\begin{cases} y^{(n)} &= f(t, y, y', y'', \ldots, y^{(n-1)}) \\ y^{(j)}(t_0) &= y_j \qquad \textit{for all } j = 0, 1, \ldots, n-1 \end{cases}.$$

We have the following:

(1) *If f is continuous, then there exists an interval $a < t_0 < b$ and a function $y(t)$ such that*

$$y^{(n)}(t) = f\left(t, y(t), y'(t), y''(t), \dots, y^{(n-1)}(t)\right)$$

for all t such that $a < t < b$ and

$$y(t_0) = y_0,\ y'(t_0) = y_1,\ \cdots,\ y^{(n-1)}(t_0) = y_{n-1}.$$

(2) *Let f_k be the k^{th} partial derivative of $f(x_0, \dots, x_n)$, that is*

$$f_k(x_0, \dots, x_n) = \frac{\partial f}{\partial x_k}(x_0, \dots, x_n).$$

If f_k for $k = 0, 1, \dots, n$ are continuous then the solution that exists from (1) is unique as well.

Remark 11.2. What this theorem says in English is that if f is nice, then there exists solutions for a small amount of time (but there could be many). If f is even nicer, then there exists a unique solution for a small amount of time.

Example 11.2. This is an example of a differential equation which does not have a unique solution.

Consider the initial value problem:

$$\begin{cases} y' &= y^{1/3} \\ y(0) &= 0 \end{cases}.$$

Here are infinitley many solutions to the differential equation:

$$y(t) = 0 \qquad \text{and} \qquad y(t) = \pm\left(\frac{2}{3}(t-a)\right)^{3/2} \text{ where } a \geq 0.$$

Now we discuss (briefly) what complex numbers are.

A complex number z is a pair of real numbers x and y and is written as $z = x + iy$, where i is a square root of -1. Complex numbers behave (almost) exactly like real numbers except now there is a square root of -1. For example, here are some properties of complex numbers

$$\begin{aligned} \text{addition} \quad & (a+ib) + (c+id) = (a+c) + i(b+d) \\ \text{multiplication} \quad & (a+ib)\cdot(c+id) = (ac-bd) + i(bc+ad) \\ \text{exponentiation} \quad & e^z e^w = e^{z+w}. \end{aligned}$$

When $z = x + iy$, we call x the real part of z and we write $x = \text{Re}(z)$ and we call y the imaginary part of y and write $y = \text{Im}(z)$.

Since $e^{x+iy} = e^x e^{iy}$ we have to figure out what e^{iy} means. Formally, lets try to figure out what e^{iy} should be.

12. Lecture 12

Example 12.1. What is the solution of $y'' + y = 0$?

Solution: Formally, the solution is $c_1e^{r_1t} + c_2e^{r_2t}$ where $r_1 = \sqrt{-1}$ and $r_2 = -\sqrt{-1}$. That tells us the solution is

$$y(t) = c_1e^{it} + c_2e^{-it}.$$

What are some other solutions to this differential equation?

Well

$$\frac{d^2}{dt^2}\sin(t) = -\sin(t) \qquad \text{and} \qquad \frac{d^2}{dt^2}\cos(t) = -\cos(t).$$

That means we should be able to find two more constants a_1 and a_2 to say that

$$y(t) = a_1\sin(t) + a_2\cos(t)$$

By the uniqueness theorem, we must be able to write

$$a_1\sin(t) + a_2\cos(t) = c_1e^{it} + c_2e^{-it}.$$

If we set $y(0) = 1$ and $y'(0) = 0$ then the solution is equal to $\cos(t)$. And $e^{i0} = e^{-i0} = e^0 = 1$, meaning $c_1 + c_2 = 1$. And if we differential $e^{\pm it}$ we get $\pm ie^{\pm it}$. That means $ic_1 - ic_2 = 0$ and so $c_1 - c_2 = 0$. This tells us that $c_1 = c_2 = \frac{1}{2}$. This tells us that

$$\cos(t) = \frac{e^{it} + e^{-it}}{2}.$$

We can differentiate everything to get

$$-\sin(t) = \frac{ie^{it} - ie^{-it}}{2}.$$

That means

$$\sin(t) = \frac{e^{it} - e^{-it}}{2i}.$$

Therefore, we get

$$e^{it} = \frac{e^{it} + e^{-it}}{2} + i\frac{e^{it} - e^{-it}}{2i} = \cos(t) + i\sin(t).$$

□

Now we can discuss polar coordinates. Suppose I have $z = x + iy$ and I want to write that as $re^{i\theta}$ then what are r and θ? Well, we set $r = \sqrt{x^2 + y^2}$

and $\theta = \arctan(y/x)$ which work. This works when $x > 0$, a different θ pops up when $x < 0$. In particular, we can write

$$\theta = \pi - \arctan\left(-y/x\right), \qquad x < 0.$$

Before continuing let's compute a complex inverses

$$\frac{1}{a+ib} = \frac{1}{a+ib} \cdot \frac{a-ib}{a-ib} = \frac{a-ib}{a^2+b^2}.$$

Now consider the differential equation

$$\begin{cases} y'' + 2y' + 2 & = 0 \\ y(0) = 2 & \\ y'(0) = 4 & \end{cases}.$$

Then we must find the roots of

$$x^2 + 2x + 2 = 0.$$

That means

$$x = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i.$$

Thus the solutions are of the form:

$$y(t) = c_1 e^{-(1+i)t} + c_2 e^{-(1-i)t} = a_1 e^{-t}\cos(t) + a_2 e^{-t}\sin(t).$$

Differentiating this we get

$$y'(t) = -e^{-t}\left(a_1\cos(t) + a_2\sin(t)\right) + e^{-t}\left(-a_1\sin(t) + a_2\cos(t)\right).$$

The left term on the right-hand side is just $-y(t)$.

Thus

$$y(0) = a_1 = 2$$

and

$$4 = y'(0) = -y(0) + e^{-0}(a_2) = -2 + a_2$$

and so $a_2 = 6$. Thus the solution is

$$y(t) = e^{-t}\left(2\cos(t) + 6\sin(t)\right).$$

Let's try to solve this initial value problem

$$\begin{cases} 16y'' - 8y' + 145y & = 0 \\ y(0) & = -2 \\ y'(0) & = 1 \end{cases}.$$

Well the characteristic equation is

$$16r^2 - 8r + 145 = 0.$$

The roots are $\frac{1}{4} \pm 3i$. Therefore, the general solution is

$$c_1 e^{(\frac{1}{4}+3i)t} + c_2 e^{(\frac{1}{4}-3i)t} = e^{t/4}\left(a_1 \cos(3t) + a_2 \sin(3t)\right).$$

We compute

$$y'(t) = \frac{1}{4} y(t) + e^{t/4}\left(-3a_1 \sin(3t) + 3a_2 \cos(3t)\right).$$

Using $y(0) = -2$ we get $a_1 = -2$. And using $y'(0) = 1$ we get

$$1 = y'(0) = -\frac{1}{2} + (3a_2)$$

and so $a_2 = \frac{1}{2}$.

Thus the solution is

$$y(t) = -2e^{t/4} \cos(3t) + \frac{1}{2} e^{t/4} \sin(3t).$$

13. Lecture 13

Let's review the complex roots situation. Suppose we have a differential equation of the form:

$$ay'' + by' + cy = 0,$$

such that the discriminant $b^2 - 4ac < 0$. That means there are two complex roots

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \lambda \pm i\mu \qquad \text{(say)}.$$

This means that the general solution looks like

$$y(t) = c_1 e^{(\lambda + i\mu)t} + c_2 e^{(\lambda - i\mu)t}.$$

In general these c_1 and c_2 will be such that the imaginary part $y(t)$ is zero. Alternatively, we can write

$$y(t) = a_1 e^{\lambda t} \cos(\mu t) + a_2 e^{\lambda t} \sin(\mu t).$$

In general, the sin and cos formula is easier to deal with when solving initial value problems because there are fewer complex numbers to keep track of.

We now recall that we know of to solve $ay'' + by' + cy = 0$ when $b^2 - 4ac \neq 0$. We now discuss the situation when we have a repeated root, i.e. $b^2 - 4ac = 0$. This means the characteristic equation can be written as

$$a(r - \lambda)^2 = 0, \qquad \text{for some real number } \lambda.$$

(You should try to prove why λ has to be a real number and not a complex number.)

We claim that the general solution to the differential equation is

$$y(t) = a_1 e^{\lambda t} + a_2 t e^{\lambda t}.$$

In order to demonstrate this lets use the existence and uniqueness theorems. Let's consider the initial value problem:

$$\begin{cases} y'' - 2\lambda y' + \lambda^2 y &= 0 \\ y(0) &= y_0 \\ y'(0) &= y_1 \end{cases}.$$

The characteristic equation is $(r - \lambda)^2 = 0$ and so we claim the solution is the one shown above. We first rewrite y as

$$y(t) = e^{\lambda t}(a_1 + a_2 t).$$

Differentiating gives

$$y'(t) = \lambda y(t) + a_2 e^{\lambda t}$$

and

$$y''(t) = \lambda y'(t) + a_2 \lambda e^{\lambda t} = \lambda^2 y(t) + 2\lambda a_2 e^{\lambda t}$$

Thus we have

$$\begin{aligned} y'' - 2\lambda y' + \lambda^2 y &= \left(\lambda^2 y + 2\lambda a_2 e^{\lambda t}\right) - 2\lambda \left(\lambda y + a_2 e^{\lambda t}\right) + \lambda^2 y \\ &= (\lambda^2 - 2\lambda^2 + \lambda^2) y + (2\lambda a_2 - 2\lambda a_2) e^{\lambda t} = 0. \end{aligned}$$

Now we have to find an a_1 and a_2 that work. We note that

$$y_0 = y(0) = a_1$$

and

$$y_1 = y'(0) = \lambda y(0) + a_2 = \lambda y_0 + a_2,$$

thus the specific solution is

$$y(t) = y_0 e^{\lambda t} + (y_1 - \lambda y_0) t e^{\lambda t}.$$

This is unique as well.

Example 13.1. Solve the initial value problem

$$\begin{cases} y'' + 2y' + y &= 0 \\ y(0) &= 2 \\ y(1) &= 3 \end{cases}.$$

***Solution*:** The characteristic equation is

$$r^2 + 2r + 1 = (r+1)^2 = 0.$$

Thus the solution should be

$$y(t) = a_1 e^{-t} + a_2 t e^{-t}.$$

We find $a_1 = y(0) = 2$ and

$$3 = y(1) = 2e^{-1} + a_2 e^{-1} = \frac{2 + a_2}{e}$$

Thus $a_2 = 3e - 2$. This gives the solution as

$$y(t) = 2e^{-t} + (3e - 2) t e^{-t}.$$

□

14. Lecture 14

Before we turn to the damped harmonic oscillator problem, we'll state something that is useful for the homework. That is the definitions of sinh and cosh. Specifically

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$
$$\cosh(x) = \frac{e^x + e^{-x}}{2}.$$

Let's consider the damped harmonic oscillator again without external forcing. (See the discussion during lecture 11 for more details about how to set up the differential equation.) We have the following second-order differential equation:

$$mu'' + \gamma u' + ku = -mg,$$

where $m > 0$, and $\gamma, k \geq 0$. As a method of substitution (which we have not covered in detail) we can consider the function

$$x(t) = u(t) + \frac{mg}{k}.$$

Then

$$x' = u' \qquad \text{and} \qquad x'' = u''.$$

We can then write

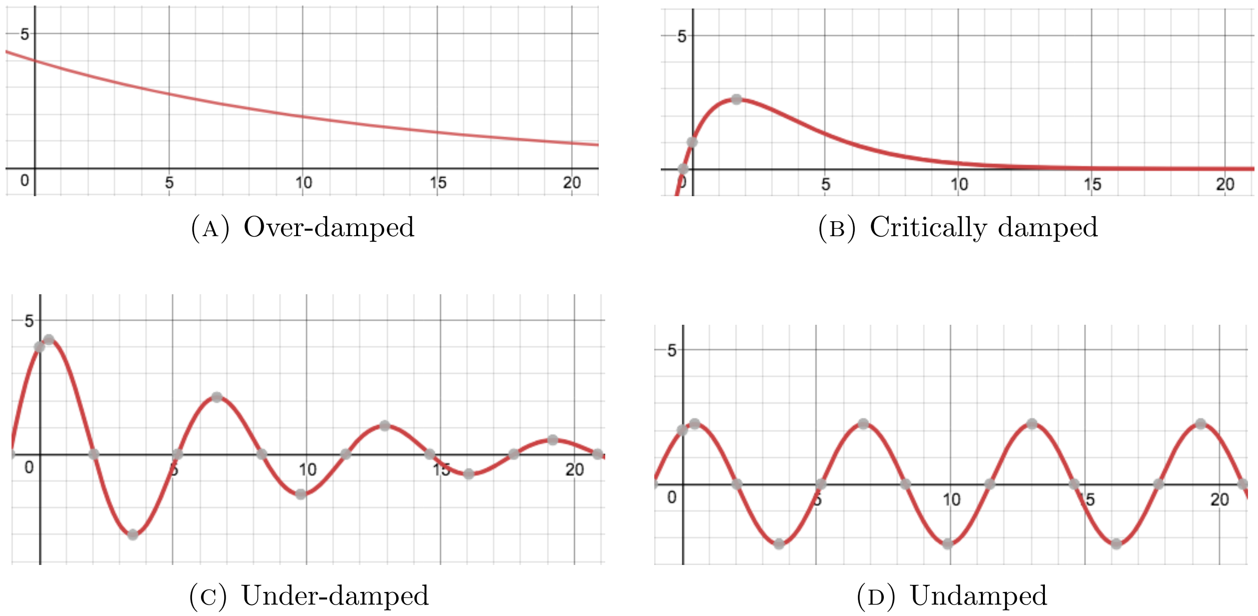
$$mx'' + \gamma x' + kx = mu'' + mu' + ku + k\frac{mg}{k} = 0.$$

We can solve this equation.

Let's study what happens to solutions based on the value of $\gamma^2 - 4mk$.

- $\gamma^2 - 4mk > 0$ corresponds to the **over-damped** spring system. Since $\gamma \geq 0$ (and in this situation $\gamma > 0$) we have the roots to the characteristic equation are distinct and negative.
- $\gamma^2 - 4mk = 0$ corresponds to the **critically damped** spring system. There is a repeated negative root.
- $\gamma^2 - 4mk < 0$ and $\gamma > 0$ corresponds to the **under-damped** spring system. The roots are complex conjugates of each other and the real part is negative.
- $\gamma = 0$ corresponds to the **un-damped** spring system. The roots are purely imaginary.

For those interested there is a similar phenomena which occurs in a closed circuit, which relies on Kirchhoff's law.



(A) Over-damped (B) Critically damped (C) Under-damped (D) Undamped

FIGURE 6. Types of damping

Let's consider the $\gamma^2 - 4mk < 0$ case where $\gamma > 0$. In this situation, we can write

$$\lambda = \frac{-\gamma}{2m} \qquad \text{and} \qquad \omega_0 = \frac{\sqrt{|\gamma^2 - 4mk|}}{2m},$$

so that the roots are

$$r_1, r_2 = \lambda \pm i\omega_0.$$

That means the solution can be written as

$$x(t) = e^{\lambda t}\left(A\cos(\omega_0 t) + B\sin(\omega_0 t)\right).$$

We recall the angle sum formula which says:

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta).$$

In particullar, if δ is a number then for all t,

$$\cos(\omega_0 t - \delta) = \cos(\omega_0 t)\cos(\delta) + \sin(\omega_0 t)\sin(\delta).$$

So if $R > 0$ and $R\cos\delta = A$ and $R\sin\delta = B$ then we can re-write the solution $x(t)$ as

$$x(t) = Re^{\lambda t}\cos(\omega_0 t - \delta) = Re^{\lambda t}\cos\left(\omega_0(t - t_0)\right).$$

This form may be more familiar to those who have taken a physics class.

15. Lecture 15

We'll now move onto **non-homogeneous constant coefficient second-order linear differential equations.** We have not discussed (in general) what non-homogeneous differential equations are so don't worry if you don't know what that word means.

So far we have worked on differential equations of the form (where I define $L[y]$ below):

$$L[y] := ay'' + by' + cy = 0.$$

We are going to call the general solution to this type of equation the homogeneous solution and denote that by y_h. In particular, we have

$$y_h(t) = \begin{cases} c_1 e^{r_1 t} + c_2 e^{r_2 t} & \text{distinct real roots} \\ e^{\lambda t}\left(c_1 \cos(\mu t) + c_2 \sin(\mu t)\right) & \text{distinct complex roots} \\ c_1 e^{rt} + c_2 t e^{rt} & \text{repeated root} \end{cases},$$

and

$$L[y_h] = 0.$$

The non-homogeneous problem is of the form

$$L[y] = f(t),$$

and then we want to find a particular solution $y_p(t)$ such that $L[y_p] = f(t)$. If we can find such a y_p, then we can solve any initial value problem involving the differential equation $L[y] = f(t)$, and the solution will be of the form

$$y(t) = y_p(t) + y_h(t).$$

There are three ways to find these particular solutions and they are

- Method of Undetermined Coefficients
- Laplace Transforms
- Variation of Parameters (not covered in this class)

We'll discuss now, the method of undetermined coefficients. This works in the case where the function $f(t)$ is of the form

$$f(t) = p(t)e^{rt} \qquad \text{or} \qquad f(t) = e^{rt}\left(p(t)\sin(\omega t) + q(t)\cos(\omega t)\right)$$

where p and q are polynomials and r and ω are numbers. It also works in the case where f is the sum of things of the above form.

Here are some examples of differential equations where undetermined coefficients will work

$$\begin{aligned} y'' + 2y' - 6y &= (2-t)e^{25t} \\ y'' - 2y' + y &= \sin(3t) + \cos(4t) \\ y + 2y'' - 2y' &= e^t \sin(t) \end{aligned}$$

Let's do a particular example.

Example 15.1. Find the particular solution y_p to the differential equation

$$y'' + 3y' + 2y = (t-2)e^{2t}.$$

***Solution*:** We notice that $f(t) = (t-2)e^{2t}$ is of the desired form. The polynomial is $p(t) = t - 2$, which is of degree 1, and the exponential is e^{2t}. We guess that the solution to this differential equation is of the form $y_p(t) = (At + B)e^{2t}$, that is we take a degree 1 polynomial, and multiply it by e^{2t}.

Let's differentiate y_p. We get

$$\begin{aligned} y_p' &= Ae^{2t} + 2(At+B)e^{2t} \\ &= e^{2t}\left(2At + A + 2B\right) \\ y_p'' &= 2Ae^{2t} + 2Ae^{2t} + 4(At+B)e^{2t} \\ &= e^{2t}\left(4At + 4A + 4B\right) \end{aligned}$$

and so

$$y_p'' + 3y_p' + y_p = e^{2t}\left(12At + 7A + 12B\right),$$

and we want this to be

$$(t-2)e^{2t}.$$

That means

$$12A = 1 \qquad \text{and} \qquad 7A + 12B = -2.$$

This tells us that $A = \frac{1}{12}$ and $B = \frac{-31}{144}$. That tells us the particular solution is

$$y_p(t) = \left(\frac{1}{12}t - \frac{31}{144}\right)e^{2t}.$$

□

Example 15.2. Find the particular example of the differential equation

$$y'' + 3y' + 2y = (t^2-2)e^{2t}.$$

Solution: Since $f(t) = p(t)e^{rt}$ when p is a degree 2 polynomial, we expect the particular solution to be

$$y_p(t) = (At^2 + Bt + C)e^{2t}.$$

We now compute derivatives

$$\begin{aligned}
y_p &= e^{2t}\left(At^2 + Bt + C\right) \\
y_p' &= (2At + B)e^{2t} + 2(At^2 + Bt + C)e^{2t} \\
&= e^{2t}\left(2At^2 + (2A + 2B)t + (B + 2C)\right) \\
y_p'' &= 2e^{2t}\left(2At^2 + (2A + 2B)t + (B + 2C)\right) + e^{2t}\left(4At + 2A + 2B\right) \\
&= e^{2t}\left(4At^2 + (8A + 4B)t + (2A + 4B + 4C)\right).
\end{aligned}$$

That means we have $y_p'' + 3y_p' + 2y_p$ is

$$e^{2t}\left(12At^2 + (14A + 12B)t + (2A + 7B + 12C)\right) = e^{2t}(t^2 - 2).$$

Comparing coefficients is

$$\begin{aligned}
12A &= 1 \\
14A + 12B &= 0 \\
2A + 7B + 12C &= -2,
\end{aligned}$$

that means

$$\begin{aligned}
A &= \frac{1}{12} \\
B &= \frac{-14}{144} = \frac{-7}{72} \\
C &= \frac{-107}{864}.
\end{aligned}$$

That means the paritcular solution is

$$y_p(t) = \left\{\frac{1}{12}t^2 - \frac{7}{72}t - \frac{107}{864}\right\} e^{2t}.$$

□

Example 15.3. Find the particular solution to:

$$y'' + y' + y = \cos(t).$$

***Solution*:** This one is a little different than the exponential case. If we guess the particular solution is of the form

$$y_p(t) = A\cos(t)$$

we end up with

$$y_p'' + y_p' + y_p = -A\cos(t) - A\sin(t) + A\cos(t) = -A\sin(t),$$

which is not of the correct form. That means we have to look at a function of the form

$$y_p(t) = A\cos(t) + B\sin(t).$$

In this case

$$y_p'(t) = B\cos(t) - A\sin(t) \qquad \text{and} \qquad y_p''(t) = -A\cos(t) - B\sin(t).$$

That means

$$y_p'' + y_p' + y_p = B\cos(t) - A\sin(t),$$

and we want this to be

$$A\sin(t) = 0 \qquad \text{and} \qquad B\cos(t) = \cos(t).$$

Setting $A = 0$ and $B = 1$ gives the result.

So the particular solution in this case is $y_p(t) = \sin(t)$. $\square$

In general in order to solve these types of problems you have to be patient with the algebra and careful while differentiating. Here is a rough table of the form of the right-hand side and the guess for a particular solution.

$f(t)$	y_p
e^{rt}	Ae^{rt}
$\sin(at)$ or $\cos(at)$	$A\sin(at) + B\cos(at)$
$\displaystyle\sum_{k=1}^{n} a_k t^k$	$\displaystyle\sum_{k=1}^{n} b_k t^k$
$\left(\displaystyle\sum_{k=1}^{n} a_k t^k\right) e^{rt}$	$\left(\displaystyle\sum_{k=1}^{n} b_k t^k\right) e^{rt}$
$\left(\displaystyle\sum_{k=1}^{n} a_k t^k\right) \cos(at)$ or $\left(\displaystyle\sum_{k=1}^{n} a_k t^k\right) \sin(at)$	$\left(\displaystyle\sum_{k=1}^{n} b_k t^k \cos(at) + c_k t^k \sin(at)\right)$
$\left(\displaystyle\sum_{k=1}^{n} a_k t^k\right) e^{rt}\cos(at)$ or $\left(\displaystyle\sum_{k=1}^{n} a_k t^k\right) e^{rt}\sin(at)$	$e^{rt}\left(\displaystyle\sum_{k=1}^{n} b_k t^k \cos(at) + c_k t^k \sin(at)\right)$

16. Lecture 16

In this lecture we'll discuss in more detail the differential equation corresponding to **forced harmonic oscillators without damping**. These are second ordered non-homogeneous differential equations. That is we'll consider the differential equations of the form

$$mu'' + ku = F(t) \qquad \text{where } m > 0, k > 0.$$

We want these assumptions on k and m because otherwise it is very simple to solve. These occur in spring systems when there is no damping, or in circuits when there are no resistors.

In particular, we'll be concerned mostly about the situation when the external forcing term $F(t)$ is of the form $F(t) = F_0 \cos(\omega t)$. That means:

$$mu'' + ku = F_0 \cos(\omega t).$$

We'll solve the homogeneous solution and notice that we get

$$r = \frac{\pm\sqrt{-4mk}}{2m} = \pm i\sqrt{\frac{k}{m}},$$

and so the solution to the homogeneous equation is

$$u_h(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t), \qquad \text{where } \omega_0 = \sqrt{\frac{k}{m}}.$$

The particular solution is of the form

$$u_p(t) = A\cos(\omega t) + B\sin(\omega t).$$

We find $u_p''(t) = -\omega^2 u_p(t)$ and so

$$mu_p'' + ku_p = \left(k - \omega^2 m\right)\left(A\cos(\omega t) + B\sin(\omega t)\right),$$

which we set it to $F_0 \cos(\omega t)$.

That means we have

$$A = \frac{F_0}{k - \omega^2 m}, \qquad \text{and} \qquad B = 0.$$

But there is a problem when $k - \omega^2 m = 0$, which means $\omega^2 = \frac{k}{m}$, i.e. $\omega = \omega_0$. We'll correct that problem in a little bit. However, we do get

$$u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t) \qquad \text{when } \omega \neq \omega_0.$$

A problem occurs when $\omega = \omega_0$, in that case the external forcing $F_0 \cos(\omega_0 t)$ is a solution to the homogeneous solution. That means the particular solution is (and you can check)

$$u_p(t) = \frac{F_0}{2m\omega_0} t \sin(\omega_0 t),$$

and the solution is

$$u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{2m\omega_0} t \sin(\omega_0 t).$$

Let's analyze some of these problems.

Example 16.1. Solve the initial value problem:

$$u'' + u = -2\cos(3t), \qquad u(0) = u'(0) = 0.$$

Solution: We have $\omega_0 = 1$ and $\omega = 3$, that is the case $\omega_0 \neq \omega$. Therefore the general solution is

$$u(t) = c_1 \cos(t) + c_2 \sin(t) + \frac{1}{4}\cos(3t).$$

We get $0 = u(0) = c_1 + \frac{1}{4}$ and

$$0 = u'(0) = -c_1 \sin(0) + c_2 \cos(0) - \frac{3}{4}\sin(3t) = c_2.$$

So the solution is

$$u(t) = \frac{1}{4}\left(\cos(3t) - \cos(t)\right).$$

It turns out that we can re-write this equation in the form

$$u(t) = \left[\frac{-1}{2}\sin(t)\right]\sin(2t)$$

□

The behavior in the previous example is more general, that is if

$$mu'' + ku = F_0 \cos(\omega t), \quad u'(0) = u(0) = 0 \qquad \text{where } \omega_0 \neq \omega,$$

then the general solution is of the form

$$\begin{aligned} u(t) &= \frac{F_0}{m(\omega_0^2 - \omega^2)}\left(\cos(\omega t) - \cos(\omega_0 t)\right) \\ &= \left[\frac{2F_0}{m(\omega_0^2 - \omega^2)}\sin\left(\frac{\omega_0 - \omega}{2}t\right)\right]\sin\left(\frac{\omega_0 + \omega}{2}t\right) \end{aligned}$$

What happens when $\omega_0 - \omega$ is really small? Well, let's consider the case where

$$\omega_0 = 1 \qquad \text{and} \qquad \omega = 0.9.$$

Then the solutions is of the form:

$$C \sin(0.05t) \sin(0.95t)$$

The $\sin(0.95t)$ terms oscillates much much more than the $\sin(0.05t)$ term. Here's an example that demonstrates the behavior

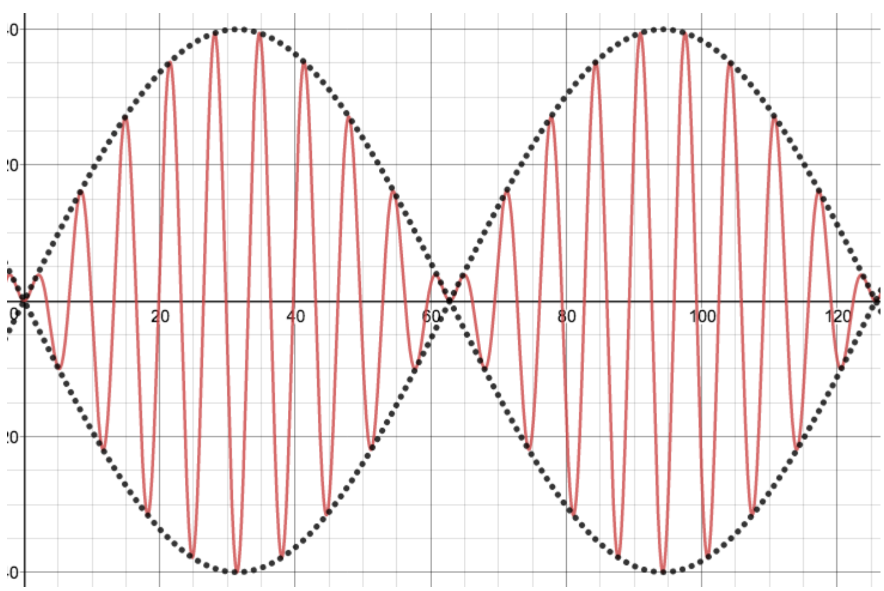


FIGURE 7. The red line is the solution and the dotted black line is the oscillations corresponding to $\sin(0.05t)$.

The same does not occur in the solutions to in the case where $\omega_0 = \omega$. That's because term $t\sin(\omega_0 t)$ appearing in the solution begins to dominate for large t. Here's a graph of what happens

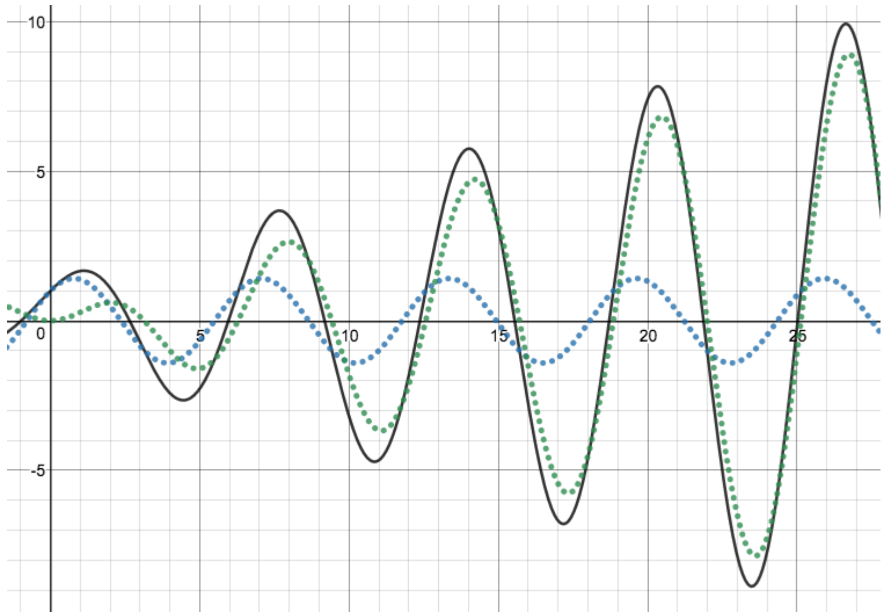


FIGURE 8. The black solid line is the curve $\sin(t)+\cos(t)+\frac{1}{3}t\sin(t)$. The blue dotted line is the curve $\sin(t)+\cos(t)$ and the green dotted line is $\frac{1}{3}t\sin(t)$.

As you can see in the figure, the green dotted line is very closely related to the black curve. That's because the $t\sin(t)$ term dominates. For larger values of t the influence of the $\sin(t)+\cos(t)$ term becomes much less influential.

Physically what's happening here is that the external forcing matches the oscillations of the spring system. So the external forcing is adding energy to the system.

17. Lecture 17

In this lecture we'll consider the differential equation corresponding to **forced harmonic oscillators with damping**. In particular, we'll want the forcing term to be periodic, and so we'll consider the differential equations of the form:

$$mu'' + \gamma u' + ku = F_0 \cos(\omega t).$$

Before describing the general behavior of these problems, we'll look at a simple example.

Example 17.1. Consider the initial value problem

$$u'' + u' + 1.25u = 3\cos t,$$

and

$$u(0) = 2, \qquad u'(0) = 3.$$

What is the behavior of the solution for large t?

***Solution*:** The characteristic equation is

$$r^2 + r + 1.25 = 0 \qquad \text{so } r = \frac{-1 \pm \sqrt{1-5}}{2} = -\frac{1}{2} \pm i.$$

So the solution to the homogeneous equation is of the form

$$u_h(t) = c_1 e^{-t/2}\cos(t) + c_2 e^{-t/2}\sin(t).$$

We guess that the particular solution is of the form

$$u_p(t) = A\cos(t) + B\sin(t),$$

and we can check that

$$u_p'' + u_p' + u_p = \left(\frac{1}{4}A + B\right)\cos(t) + \left(-A + \frac{1}{4}B\right)\sin(t).$$

We then find that $A = \frac{12}{17}$ and $B = \frac{48}{17}$ (you should check this).

That means the general solution to the differential equation is

$$c_1 e^{-t/2}\cos(t) + c_2 e^{-t/2}\sin(t) + \frac{12}{17}\cos(t) + \frac{48}{17}\sin(t).$$

Solving the initial value problem gives

$$u(0) = c_1 + \frac{12}{17} = 2 \qquad \text{and} \qquad u'(0) = -\frac{1}{2}c_1 + c_2 + \frac{48}{17} = 3,$$

and

$$u(t) = \frac{22}{17}e^{-t/2}\cos(t) + \frac{14}{17}e^{-t/2}\sin(t) + \frac{12}{17}\cos(t) + \frac{48}{17}\sin(t).$$

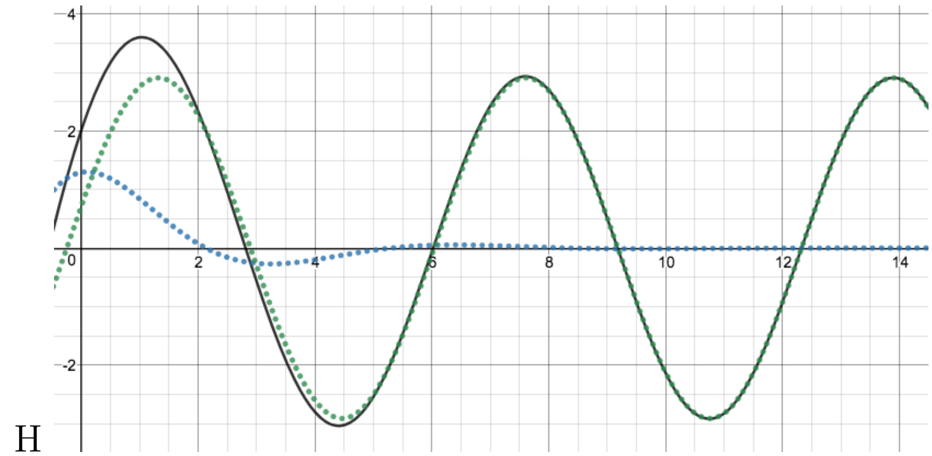


FIGURE 9. The black line is $u(t)$, the dotted blue line is $u_h(t)$ and the green dotted line is $u_p(t)$.

We note for large t, the $e^{-t/2}$ term is really, really small so the dominating term is the $u_p(t)$. Here's the graph

We notice that for t as small as 8 the graph of $u_p(t)$ is practically the graph of the actual solution. □

Let's move to the general situation. That is we have the situation where

$$mu'' + \gamma u' + ku = F_0 \cos(\omega t) \qquad \text{where } m, \gamma > 0, k \geq 0.$$

The roots of the characteristic equation are

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

and we can have various situations as outlined in lecture 14.

The general solution is of the form

$$u(t) = u_h(t) + u_p(t).$$

Since $\gamma \neq 0$ we either get an over-damped system, critically damped system or under-damped system. In any of those cases

$$\lim_{t \to \infty} u_h(t) = 0.$$

Physically that means the influence of the homogeneous spring solution eventually has very little effect. We call the solution $u_h(t)$ a **transient solution**, because (depending on γ) the effect of $u_h(t)$ becomes undetectable very quickly.

In contrast, the $u_p(t)$ never dies out, and so we call this the **steady state solution.** In practice, it is very useful to express $u_p(t)$ in terms of a single trigonometric expression. We can write

$$u_p(t) = A\cos(\omega t) + B\sin(\omega t),$$

where we can find A and B by using the differential equation. Using various trig identities, you can show that

$$u_p(t) = R\cos(\omega t - \delta),$$

where

$$R = \frac{F_0}{\Delta}, \qquad \cos(\delta) = \frac{m(\omega_0^2 - \omega^2)}{\Delta}, \qquad \sin(\delta) = \frac{\gamma\omega}{\Delta}$$

for

$$\Delta = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} \qquad \text{and} \qquad \omega_0^2 = \frac{k}{m}.$$

We can re-write

$$R = \frac{F_0}{k\sqrt{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \frac{\gamma^2\omega^2}{mk\omega_0^2}}}.$$

Observe that as $\omega \to 0$ the term R approaches $\frac{F_0}{k}$ and as $\omega \to \infty$ the quantity $R \to 0$.

18. Lecture 18

Let's review how to give meaning to the improper integral:

$$\int_a^\infty f(t)\,dt.$$

If for each $A > a$ the integral

$$\int_a^A f(t)\,dt$$

exists AND the limit

$$\lim_{A\to\infty}\int_a^A f(t)\,dt \qquad \text{exists,}$$

then we define

$$\int_a^\infty f(t)\,dt = \lim_{A\to\infty}\int_a^A f(t)\,dt$$

and we say that the integral $\int_a^\infty f(t)\,dt$ **converges**. If that limit does not exist, or $\int_a^A f(t)\,dt$ doesn't exist for some A then we say that the improper integral diverges.

Example 18.1. Here are some improper integrals:

(1) If $c \neq 0$ then

$$\begin{aligned}\int_0^\infty e^{ct}\,dt &= \lim_{A\to\infty}\int_0^A e^{ct}\,dt\\ &= \lim_{A\to\infty}\frac{1}{c}e^{ct}\Big|_0^A\\ &= \lim_{A\to\infty}\frac{1}{c}(e^{cA}-1) = \begin{cases}\infty & c>0\\ \frac{-1}{c} & c<0\end{cases}\end{aligned}$$

(2) If $p > 0$ then

$$\begin{aligned}
\int_1^p \frac{1}{t^p}\,dt &= \lim_{A\to\infty}\int_1^A t^{-p}\,dt \\
&= \lim_{A\to\infty} \frac{1}{1-p} t^{1-p}\Big|_1^A \\
&= \lim_{A\to\infty} \frac{A^{1-p}-1}{1-p} \\
&= \begin{cases} \infty & p \le 1 \\ \frac{1}{p-1} & p > 1 \end{cases}.
\end{aligned}$$

In general, it is difficult to compute integrals explicitly. Sometimes we want to know whether an improper integral exists even when computing a definite example is difficult or impossible, but we still want to know whethe an integral converges. For example consider

$$\int_0^\infty e^{-x^2}\,dx.$$

Does that integral converge? It turns out that it does, but it's impossible to actually find an explicit form for the integral

$$\int_0^A e^{-x^2}\,dx.$$

We do have this theorem which is useful:

Theorem 18.1. *Suppose that f is a piecewise continuous function for $t \geq a$.*

(1) *If $|f(t)| \leq g(t)$ for all $t \geq M$, then*

$$\int_a^\infty g(t)\,dt \text{ converges implies that } \int_a^\infty f(t)\,dt \text{ converges.}$$

(2) *If $f(t) > g(t)$ for all $t \geq M$, then*

$$\int_a^\infty g(t)\,dt \text{ diverges implies that } \int_a^\infty f(t)\,dt \text{ diverges.}$$

Remark 18.1. The good functions for comparison for the above theorem are e^{ct} and t^{-p}.

We will now discuss **integral transforms**. In this context an integral transform is integration against some **kernel** $K(s,t)$, that is we focus on:

$$F(s) = \int_{\alpha}^{\beta} K(s,t) f(t)\, dt.$$

In particular, we can look at $K(s,t) = e^{-st}$ and $\alpha = 0$ and $\beta = \infty$. That is, we'll define the **Laplace transform** of f by the function of s defined by:

$$\mathcal{L}(f)(s) = \int_0^{\infty} e^{-st} f(t)\, dt.$$

Example 18.2. Let's compute the Laplace transform of several functions.

(1) Let's try $f(t) = 1$.

$$\mathcal{L}(1)(s) = \int_0^{\infty} e^{-st}\, dt = \begin{cases} \infty & s \le 0 \\ \frac{1}{s} & s > 0. \end{cases}.$$

(2) Let's try $f(t) = t$, for $s > 0$ we get

$$\begin{aligned}
\mathcal{L}(t)(s) &= \int_0^{\infty} e^{-st} t\, dt \\
&= \lim_{A\to\infty} \int_0^{A} e^{-st} t\, dt \\
&= \lim_{A\to\infty} \left.\frac{te^{-st}}{-s}\right|_{t=0}^{t=A} - \int_0^{A} \frac{e^{-st}}{-s}\, dt \\
&= \lim_{A\to\infty} \frac{Ae^{-sA}-s}{-s} - \left.\frac{e^{-st}}{s^2}\right|_{t=0}^{t=A} \\
&= \lim_{A\to\infty} \frac{Ae^{-sA}}{-s} - \frac{e^{-sA}}{s^2} + \frac{1}{s^2} \\
&= \begin{cases} \infty & s \le 0 \\ \frac{1}{s^2} & s > 0. \end{cases}
\end{aligned}$$

(3) Let's try $f(t) = e^{at}$ for some number a.

$$\begin{aligned}
\mathcal{L}(e^{at})(s) &= \int_0^{\infty} e^{-st} e^{at}\, dt = \int_0^{\infty} e^{(a-s)t}\, dt \\
&= \begin{cases} \frac{1}{s-a} & s > a \\ \infty & s \le a \end{cases}
\end{aligned}$$

(4) Let's try when $f(t) = \sin(at)$ for some a. For $s > 0$ we have

$$\begin{aligned}
\mathcal{L}(\sin(at))(s) &= \int_0^\infty e^{-st}\sin(at)\,dt \\
&= -\frac{1}{a}e^{-st}\cos(at)\Big|_{t=0}^{t=\infty} - \int_0^\infty \frac{s}{a}e^{-st}\cos(at)\,dt \\
&= \frac{1}{a} - \frac{s}{a}\left(\frac{1}{a}e^{-st}\sin(at)\Big|_0^\infty - \int_0^\infty \frac{-s}{a}e^{-st}\sin(at)\,dt\right) \\
&= \frac{1}{a} - \frac{s}{a}\left(\frac{s}{a}\mathcal{L}(\sin(at))(s)\right) \\
(1+\frac{s^2}{a^2})\mathcal{L}(\sin(at))(s) &= \frac{1}{a} \\
(a^2+s^2)\mathcal{L}(\sin(at))(s) &= a \\
\mathcal{L}(\sin(at))(s) &= \frac{a}{s^2+a^2}.
\end{aligned}$$

(5) Try to show that

$$\mathcal{L}(\cos(at)) = \frac{s}{a^2+s^2}.$$

The previous theorem is useful if we want to know when a Laplace transform exists:

Theorem 18.2. *Suppose that f is piece-wise continuous function for all $t \geq 0$ such that there exists some constant C and a such that $|f(t)| \leq Ke^{at}$ for all $t \geq M$. Then*

$$\mathcal{L}(f)(s) \text{ exists for all } s > a.$$

Question: Why would we look at this Laplace transform when this is a class on differential equations?

Answer: Consider a function f which is differentiable and has derivative f'. Then integration by parts tells us the following

$$\begin{aligned}
\int_0^A e^{-st}f'(t)\,dt &= f(t)e^{-st}\Big|_{t=0}^{t=A} - \int_0^A -se^{-st}f(t)\,dt \\
&= f(A)e^{-sA} - f(0) + s\int_0^A e^{-st}f(t)\,dt.
\end{aligned}$$

If we can take the limits as $A \to \infty$ we can say that

$$\mathcal{L}(f')(s) = s\mathcal{L}(f)(s) - f(0)$$

provided that $\mathcal{L}(f)(s)$ exists and $f(A)e^{-sA} \to 0$ as $A \to \infty$.

19. Lecture 19:

Let's start today's lecture with an important formula. It may not seem important now, but hopefully by the end of the class it will seem more useful. It is the *exponential shift formula*. Suppose that $\mathcal{L}(f(t))(s) = F(s)$, then

$$\mathcal{L}(e^{at}f(t))(s) = F(s-a).$$

Indeed,

$$\begin{aligned} F(s-a) = \mathcal{L}(f(t))(s-a) &= \int_0^\infty e^{-(s-a)t} f(t)\,dt \\ &= \int_0^\infty e^{-st}e^{at} f(t)\,dt = \mathcal{L}(e^{at}f(t))(s). \end{aligned}$$

We finished last lecture with (part of) the statement of theorem

Theorem 19.1. *Suppose that f has Laplace transform $F(s)$. If f' is continuous then*

$$\mathcal{L}(f'(t))(s) = sF(s) - f(0).$$

If f'' is continuous then we can say

$$\mathcal{L}(f''(t))(s) = s^2F(s) - sf(0) - f'(0).$$

Now we look at the second order differential equation

$$ay'' + by' + cy = f(t).$$

If $\mathcal{L}(y)(s) = Y(s)$ and $\mathcal{L}(f)(s) = F(s)$ then we can write

$$\begin{aligned} a\mathcal{L}(y'')(s) + b\mathcal{L}(y')(s) + c\mathcal{L}(y)(s) &= \mathcal{L}(f)(s) \\ a\left(s^2Y(s) - sy(0) - y'(0)\right) + b\left(sY(s) - y(0)\right) + cY(s) &= F(s) \\ \left(as^2 + bs + c\right)Y(s) - \left(asy(0) + ay'(0) + by(0)\right) &= F(s) \\ \frac{F(s)}{as^2+bs+c} + \frac{asy(0) + ay'(0) + by(0)}{as^2+bs+c} &= Y(s). \end{aligned}$$

If we can undo the Laplace transform then we can recover a solution y which solves an initial value problem.

It turns out that in a lot of circumstances finding the inverse Laplace transform is possible. If $\mathcal{L}(f(t))(s) = F(s)$ then we write $\mathcal{L}^{-1}(F(s))(t) = f(t)$.

Computing inverse Laplace transforms often involves using partial fraction decompositions and then using the Laplace transform table found in the textbook.

Example 19.1. If

$$F(s) = \frac{3s}{s^2 - s - 6},$$

then what is $f(t) = \mathcal{L}^{-1}(F)(t)$?

***Solution*:** We start by finding the partial fraction decomposition. Since $(s^2 - s - 6) = (s-3)(s+2)$ we want to find some constants A and B such that

$$\frac{3s}{s^2 - s - 6} = \frac{A}{s-3} + \frac{B}{s+2}$$

In order to do this, we multiply everything by the denominator on the left-hand side and get

$$3s = A(s+2) + B(s-3).$$

If we plug in $s = 3$ we get

$$9 = 3 \cdot 3 = A(3+2) + B(3-3) = 5A,$$

implying that $A = \frac{9}{5}$. Similarly, if $s = -2$ we get

$$-6 = B(-5) \qquad \text{so} \qquad B = \frac{6}{5}.$$

This tells us that

$$F(s) = \frac{9/5}{s-3} + \frac{6/5}{s+2}.$$

Since the Laplace transform is linear, its inverse is linear as well. That means we can write:

$$\mathcal{L}^{-1}(F)(t) = \frac{9}{5}\mathcal{L}^{-1}\left(\frac{1}{s-3}\right)(t) + \frac{6}{5}\mathcal{L}^{-1}\left(\frac{1}{s+2}\right)(t).$$

But since $\mathcal{L}(e^{at})(s) = \frac{1}{s-a}$ we can say that

$$\mathcal{L}^{-1}\left(\frac{1}{s-a}\right)(t) = e^{at}.$$

Meaning that

$$f(t) = \mathcal{L}^{-1}(F)(t) = \frac{9}{5}e^{3t} + \frac{6}{5}e^{-2t}.$$

□

Example 19.2. Find the inverse Laplace transform of $F(s) = \frac{2s-3}{s^2+2s+10}$.

***Solution*:** We now that $s^2+2s+10$ only has complex roots $(2^2-4\cdot1\cdot10<0)$ and so we cannot use partial fractions. Instead, we'll complete the square. That is we want to find an a and b such that

$$(s+a)^2+b=s^2+2s+10.$$

However,

$$(s+a)^2+b=s^2+2as+a^2+b,$$

and so we can easily see that $a=1$ and $b=10-1=9$. That means we have

$$\frac{2s-3}{s^2+2s+10}=\frac{2s-3}{(s+1)^2+9}=\frac{2(s+1)-5}{(s+1)^2+9}.$$

We now want to find

$$\mathcal{L}^{-1}\left(\frac{2(s+1)}{(s+1)^2+9}\right)(t)\qquad\text{and}\qquad\mathcal{L}^{-1}\left(\frac{-5}{(s+1)^2+9}\right)(t).$$

We know that (using the exponential shift formula and the Laplace transform table)

$$\mathcal{L}(e^{bt}\cos(at))(s)=\frac{(s-b)}{(s-b)^2+a^2}.$$

That means

$$\mathcal{L}^{-1}\left(\frac{2(s+1)}{(s+1)^2+3^2}\right)(t)=2\mathcal{L}^{-1}\left(\frac{(s+1)}{(s+1)^2+3^2}\right)(t)=2e^{-t}\cos(3t).$$

We also know that

$$\mathcal{L}(e^{bt}\sin(at))=\frac{a}{(s-b)^2+a^2}.$$

That means we can say

$$\mathcal{L}^{-1}\left(\frac{-5}{(s+1)^2+9}\right)(t)=\frac{-5}{3}\mathcal{L}^{-1}\left(\frac{3}{(s+1)^2+9}\right)(t)=-\frac{5}{3}e^{-t}\sin(3t).$$

Therefore, we can say

$$\mathcal{L}^{-1}(F)(t)=2e^{-t}\cos(3t)-\frac{5}{3}e^{-t}\sin(3t).$$

□

20. Lecture 20:

Let's use the inverse Laplace transform to solve a few initial value problems.

Example 20.1. Solve the initial value problem

$$y'' - y' - 6y = 0 \qquad \text{where} \qquad y(0) = 1, \quad y'(0) = -1.$$

***Solution*:** Let $Y(s) = \mathcal{L}(y)(s)$. Then

$$\mathcal{L}(y'') = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s + 1$$

and

$$\mathcal{L}(y') = sY(s) - y(0) = sY(s) - 1.$$

Hence, we have

$$\mathcal{L}(y'' - y' - 6y) = (s^2 - s - 6)Y(s) - s + 1 - 1 = 0.$$

That implies

$$Y(s) = \frac{s}{s^2 - s - 6} = \frac{A}{s-3} + \frac{B}{s+2}.$$

To find A and B we must solve

$$s = A(s+2) + B(s-3)$$

and plugging in $s = 3$ and $s = -2$ gives

$$A = \frac{3}{5} \qquad \text{and} \qquad B = \frac{2}{5}.$$

Thus

$$Y(s) = \frac{3/5}{s-3} + \frac{2/5}{s+2}.$$

This implies

$$\mathcal{L}^{-1}(Y)(t) = \frac{3}{5}e^{3t} + \frac{2}{5}e^{-2t}.$$

□

Example 20.2. Solve the differential equation:

$$y'' + y = \sin(2t),$$

satisfying the initial value problem

$$y(0) = 2, \qquad y'(0) = 1.$$

Solution: Let's use the method of Laplace transforms to solve this differential equation, even though we know how to solve it.

Let's apply the Laplace transform, and let $Y(s) = \mathcal{L}(y)(s)$.

Since $\mathcal{L}(y') = sY(s) - y(0)$ and $\mathcal{L}(y'') = s^2Y(s) - sy(0) - y'(0)$ and so

$$\mathcal{L}\left(y'' + y\right)(s) = (s^2+1)Y(s) - sy(0) - y'(0) = \mathcal{L}(\sin(2t)) = \frac{2}{s^2+4}.$$

Thus we arive at

$$Y(s) = \frac{1}{s^2+1}\left[\frac{2}{s^2+4} + sy(0) + y'(0)\right] = \frac{1}{s^2+1}\left[\frac{2}{s^2+4} + 2s + 1\right].$$

Simplifying gives

$$Y(s) = \frac{2}{(s^2+1)(s^2+4)} + \frac{2s}{s^2+1} + \frac{1}{s^2+1}.$$

We'll focus on the first term on the right-hand side first, we have to write that as

$$\frac{2}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}.$$

That implies

$$2 = (As+B)(s^2+4)+(Cs+D)(s^2+1) = (A+C)s^3+(B+D)s^2+(4A+C)s+(4B+D).$$

That means

$$A + C = 0, \qquad B + D = 0, \qquad 4A + C = 0, \qquad 4B + D = 2.$$

We can see that $A = C = 0$. Since $B = -D$ we get $-3D = 2$ and so

$$B = \frac{2}{3} \qquad \text{and} \qquad D = \frac{-2}{3}.$$

That means we can write

$$Y(s) = \frac{5}{3}\cdot\frac{1}{s^2+1} + \frac{-1}{3}\frac{2}{s^2+4} + 2\frac{s}{s^2+1}.$$

That means that

$$y(t) = \mathcal{L}^{-1}\left(\frac{5}{3}\frac{1}{s^2+1} - \frac{1}{3}\frac{2}{s^2+4} + 2\frac{s}{s^2+1}\right) = \frac{5}{3}\sin(t) - \frac{1}{3}\sin(2t) + 2\cos(t).$$

□

Example 20.3. If $\mathcal{L}(y) = Y$ then what differential equation does Y solve for the following differential equations:

$$y'' - ty = 0 \qquad \text{and} \qquad y(0) = 1, \quad y'(0) = 0.$$

and

$$(1 - t^2)y'' - 2ty' + \alpha(\alpha + 1)y = 0, \qquad \text{and} \quad y(0) = 0, \quad y'(0) = 1.$$

***Solution*:** Let's start with the first one.
We look at the table, and observe that if $Y(s) = \mathcal{L}(y)(s)$ then

$$\mathcal{L}(-ty) = Y'(s),$$

and

$$\mathcal{L}(y'') = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s.$$

Applying the Laplace transform to the entier differential equation gives

$$\mathcal{L}(y'' - ty) = s^2Y(s) - s + Y'(s) = 0.$$

Therefore we get

$$Y' + s^2Y = s.$$

From the table, we can get $\mathcal{L}(f)(s) = F(s)$ then $\mathcal{L}((-t)^n f(t))(s) = F^{(n)}(s)$. The Laplace transform of y' is

$$\mathcal{L}(y') = sY(s) - y(0) = sY(s)$$

and

$$\frac{d}{ds}\left[\mathcal{L}(y')(s)\right] = Y(s) + sY'(s).$$

Thus

$$\mathcal{L}(-2ty')(s) = 2\left[\mathcal{L}(-ty')(s)\right] = 2\frac{d}{ds}\left[sY(s)\right] = 2Y(s) + 2sY'(s).$$

Similarly, we get

$$\mathcal{L}(y'')(s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 1.$$

We have to differentiate this twice, so

$$\begin{aligned}
\frac{d}{ds}\left[s^2Y(s)-1\right] &= 2sY(s)+s^2Y'(s)\\
\frac{d^2}{ds^2}\left[s^2Y(s)-1\right] &= \frac{d}{ds}\left[2sY(s)+s^2Y'(s)\right]\\
&= 2Y(s)+2sY'(s)+2sY'(s)+s^2Y''(s)\\
&= 2Y+4sY'+Y''.
\end{aligned}$$

Thus we get

$$\begin{aligned}
0 &= \mathcal{L}\left[(1-t^2)y''-2ty'+\alpha(\alpha+1)y\right](s)\\
&= \mathcal{L}(y'')-\mathcal{L}(t^2y'')+2\mathcal{L}(-ty')+\alpha(\alpha+1)\mathcal{L}(y)\\
&= \left[s^2Y(s)-1\right]-\left[2Y+4sY'+Y''\right]+2\left[Y+sY'\right]+\alpha(\alpha+1)Y\\
&= -s^2Y''-2sY'+\left[s^2+\alpha(\alpha+1)\right]Y-1.
\end{aligned}$$

That means

$$s^2Y''+2sY'-\left[s^2+\alpha(\alpha+1)\right]=-1$$

□

Suppose that $F(s) = \mathcal{L}(f)$ and $G(s) = \mathcal{L}(g)$. Is there a formula for the inverse Laplace transform of FG that is:

$$\mathcal{L}^{-1}(FG)(s)$$

The answer turns out to be yes, and it is given by something called a convolution.

We define the operation $*$ by

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

Then,

$$\mathcal{L}(f * g)(s) = F(s)G(s).$$

Example 20.4. Find the inverse Laplace transform of

$$\frac{1}{(s-2)(s+1)}.$$

***Solution*:** Is $F(s) = \frac{1}{s-2}$ an $G(s) = \frac{1}{s+1}$ then

$$\mathcal{L}^{-1}(F) = e^{2t} \qquad \text{and} \qquad \mathcal{L}^{-1}(G) = e^{-t}.$$

That means the inverse Laplace transform of FG is

$$\mathcal{L}^{-1}(FG) = \left(e^{2t} * e^{-t}\right) = \int_0^t e^{2\tau}e^{-(t-\tau)}\,d\tau.$$

That right-most integral is

$$\int_0^t e^{-t}e^{3\tau}\,d\tau = e^{-t}\left(\frac{1}{3}e^{3\tau}\Big|_{\tau=0}^{\tau=t}\right) = \frac{1}{3}e^{-t}\left(e^{3t}-1\right) = \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t}.$$

□

21. Lecture 21

Last time we covered how to solve initial value problems by using the Laplace transform. One downside was that we only covered how to solve initial value problems that we already knew how to solve, except this time we used the machineary of the Laplace transform.

In order to extend this to handle other forcing functions we first want to develop some tools to handle what happens to piece-wise continuous functions when we apply the Laplace transform.

Let's start with what happens to a particular piece-wise continuous function, the *Heaviside function*:

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases} \qquad \text{where } c \geq 0.$$

Well, we can apply the Laplace transform to u_c and get

$$\mathcal{L}(u_c(t))(s) = \int_0^\infty e^{-st} u_c(t)\, dt = \int_c^\infty e^{-st}\, dt = \frac{1}{s} e^{-cs}.$$

Example 21.1. Write the function f as a constant plus a sum of Heaviside functions:

$$f(t) = \begin{cases} 2, & 0 \leq t < 4 \\ 4, & 4 \leq t < 7 \\ -2, & 7 \leq t < 3\pi \\ 1, & t \geq 3\pi \end{cases}.$$

***Solution*:** If we define $f_1(t) = 2$, then f agrees with f_1 for $0 \leq t < 4$. At $t = 4$, however, f jumps by 2 units, and so to account for this with f_1 we define

$$f_2(t) = 2 + 2u_4(t).$$

This new function f_2 agrees with f on the interval $0 \leq t < 7$. At $t = 7$ the function f jumps by -6 and so we can define

$$f_3(t) = 2 + 2u_4(t) - 6u_7(t),$$

and now f_3 agrees with f for $0 \leq t < 3\pi$.

At 3π the function f jumps by 3 and so we can finally write

$$f(t) = 2 + 2u_4(t) - 6u_7(t) + 3u_{3\pi}(t).$$

□

These Heaviside functions will help us compute Laplace transforms of more complicated functions. This is implicit in the next theorem, which is proved by integration by parts (which you should try on your own):

Theorem 21.1. *Let $F(s) = \mathcal{L}(f(t))(s)$. Let $c \geq 0$. Then*

$$\mathcal{L}\left(u_c(t)f(t-c)\right)(s) = e^{-cs}F(s).$$

Conversely, if $f(t) = \mathcal{L}^{-1}(F)$ then

$$u_c(t)f(t-c) = \mathcal{L}^{-1}\left(e^{-cs}F(s)\right).$$

Example 21.2. Compute the Laplace transform of the function:

$$f(t) = \begin{cases} \sin(t) & 0 \leq t < \pi/4 \\ \sin(t) + \cos(t - \pi/4) & \pi/4 \leq t \end{cases}.$$

***Solution*:** We first have to compute f as the sum of several functions multiplied by Heaviside functions.

If we define $f_1(t) = \sin(t)$ then $f(t)$ agrees with f_1 for all $t < \pi/4$.

For $t \geq \pi/4$ the function f is $\cos(t - \pi/4) + f_1(t)$. That means we can write

$$f(t) = \sin(t) + u_{\pi/4}(t)\left(\cos(t - \pi/4)\right).$$

Thus

$$\mathcal{L}(f) = \mathcal{L}(\sin(t)) + \mathcal{L}\left(u_{\pi/4}(t)\cos(t - \pi/4)\right).$$

We know that

$$\mathcal{L}(\sin(t)) = \frac{1}{s^2+1}.$$

According to the previous theorem, we get

$$\mathcal{L}(u_{\pi/4}(t)\cos(t - \pi/4)) = e^{-\pi s/4}\mathcal{L}(\cos(t)) = \frac{e^{-\pi s/4}s}{s^2+1}.$$

Thus

$$\mathcal{L}(f)(s) = \frac{1 + e^{-\pi s/4}s}{s^2+1}.$$

□

Example 21.3. Compute the inverse Laplace transform of

$$F(s) = \frac{1 - e^{-2s}}{s^2}.$$

***Solution*:** We note that the
$$\mathcal{L}^{-1}(\frac{1}{s^2}) = t.$$
So we just need to compute the inverse Laplace transform of
$$e^{-2s}/s^2.$$
By that previous theorem
$$\mathcal{L}^{-1}\left(e^{-cs}F(s)\right) = u_c(t)f(t-c) \qquad \text{where} \qquad f(t) = \mathcal{L}^{-1}(F(s)).$$
Thus we get
$$\mathcal{L}^{-1}(e^{-2s}/s^2) = u_2(t) \cdot (t-2).$$
That means
$$\mathcal{L}^{-1}(F(s)) = t - u_2(t) \cdot (t-2) = \begin{cases} t & t < 2 \\ 2 & t \geq 2 \end{cases}.$$
□

22. Lecture 22

Example 22.1. Let

$$g(t) = \begin{cases} 0 & : t < 5 \\ 1 & : 5 \le t < 20 \\ 0 & : 20 \le t \end{cases}.$$

Find a solution to the initial value problem:

$$2y'' + y' + 2y = g(t)$$

where

$$y(0) = y'(0) = 0.$$

***Solution*:** We clearly do not know how to solve this using the methods already discussed. So we'll apply the Laplace transform with $Y(s) = \mathcal{L}(y)$ and see what we get

$$\mathcal{L}(y'') = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) \qquad \mathcal{L}(y') = sY(s) - y(0) = sY(s).$$

Moreover, we can write

$$g(t) = u_5(t) - u_{20}(t),$$

and so

$$\mathcal{L}(g) = \frac{1}{s}\left(e^{-5s} - e^{-20s}\right).$$

Combing this we can write

$$(2s^2 + s + 2)Y(s) = \frac{e^{-5s} - e^{-20s}}{s},$$

and so

$$Y(s) = \frac{e^{-5s} - e^{-20s}}{s(2s^2 + s + 2)} = (e^{-5s} - e^{-20s})H(s),$$

where

$$H(s) = \frac{1}{s(2s^2 + s + 2)}2s^2 + s + 2.$$

If we can find an $h(t) = \mathcal{L}^{-1}(H(s))(t)$ then we can write:

$$y(t) = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left((e^{-5s} - e^{-20s})H(s)\right) = u_5(t)h(t-5) - u_{20}(t)h(t-20).$$

We want to write:

$$H(s) = \frac{A}{s} + \frac{Bs + C}{2s^2 + s + 2},$$

where we can find A, B and C by solving

$$1 = A(2s^2 + s + 2) + (Bs + C) \cdot s = (2A + B)s^2 + (A + C)s + 2A$$

.
That means $A = \frac{1}{2}$, $C = -\frac{1}{2}$ and $B = -1$.
So we get

$$H(s) = \frac{1/2}{s} + \frac{-s - \frac{1}{2}}{2s^2 + s + 2}.$$

We have to complete the square on the right-most denominator and which makes

$$2s^2 + s + 2 = 2\left(s^2 + \frac{1}{2}s + 1\right) = 2\left((s + \frac{1}{4})^2 + \frac{15}{16}\right).$$

That makes

$$\begin{aligned}
\frac{-s - \frac{1}{2}}{2s^2 + s + 2} &= \frac{1}{2} \cdot \frac{-s - \frac{1}{2}}{\left(s + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2} \\
&= \frac{-1}{2} \frac{s + \frac{1}{4}}{\left(s + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2} - \frac{1}{2} \frac{\frac{1}{4}}{\left(s + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2} \\
&= -\frac{1}{2} \frac{s + \frac{1}{4}}{\left(s + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2} - \frac{1}{2\sqrt{15}} \cdot \frac{\frac{\sqrt{15}}{4}}{\left(s + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2}.
\end{aligned}$$

Thus

$$H(s) = \frac{1}{2}\left[\frac{1}{s} - \frac{s + \frac{1}{4}}{\left(s + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2} - \frac{1}{\sqrt{15}} \cdot \frac{\frac{\sqrt{15}}{4}}{\left(s + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2}\right],$$

and so

$$\mathcal{L}^{-1}(H(s)) = \frac{1}{2}\left[1 - e^{-t/4}\cos\left(\frac{\sqrt{15}}{4}t\right) - \frac{1}{\sqrt{15}}e^{-t/4}\sin\left(\frac{\sqrt{15}}{4}t\right)\right]$$

This allows us to write a (fairly complicated) explicit solution to that differential equation. □

Remark 22.1. There is, as is usually the case in mathematics, another way to solve this problem. We could solve three separate initial value problems with the differential equations:

$$2y_j'' + y_j' + 2y_j = 0 \qquad \text{for} \quad j = 1, 3$$
$$2y_2'' + y_2' + 2y_2 = 1$$

and the initial values are defined iteratively as follows:

$$y_1(0) = 0 \qquad\qquad y_1'(0) = 0$$
$$y_2(5) = y_1(5) \qquad\qquad y_2'(5) = y_1'(5)$$
$$y_3(20) = y_2(20) \qquad\qquad y_3'(20) = y_2'(20).$$

Then if we define the function

$$y(t) = \begin{cases} y_1(t) & 0 \leq t < 5 \\ y_2(t) & 5 \leq t < 20 \\ y_3(t) & 20 \leq t \end{cases},$$

the function y satisfies the differential equation prescribed in the previous example.

This comment from the text is very useful:

> Although it may be helpful to visualize the solution ... composed of solutions of three separate initial value problems in three separate intervals, it is somewhat tedious to find the solution by solving these separate problems. Laplace transform methods provide a much more convenient and elegant approach to this problem and to others having discontinuous forcing functions.

Example 22.2. Consider the differential equation:

$$y'' + 4y = g(t), \qquad y'(0) = y(0) = 0,$$

where

$$g(t) = \begin{cases} 0 & 0 \leq t < 5 \\ \frac{t-5}{5} & 5 \leq t < 10 \\ 1 & t \geq 10. \end{cases}.$$

Find the solution $y(t)$.

***Solution*:** The easiest way we know how to solve this is by Laplace transforms. So in order to do this we must find $g(t)$ as a sum of functions multiplied by Heaviside functions.

If $g_1(t) = u_5(t) \cdot \left(\frac{t-5}{5}\right)$ then g_1 agrees with g on $[0, 10]$ (where we include 10 because the function g is actually continuous). Afterwards, $g(t)$ is the constant value 1, so if we consider

$$g_2(t) = u_5(t) \cdot \left(\frac{t-5}{5}\right) + u_{10}(t) \cdot \left(1 - \frac{t-5}{5}\right) = u_5(t) \cdot \left(\frac{t-5}{5}\right) - u_{10}(t) \cdot \left(\frac{t-10}{5}\right)$$

Then we can check that g_2 agrees with g for all $t \geq 0$.

If $f(t) = t$ then we can write

$$g(t) = \frac{1}{5}\left[u_5(t)f(t-5) - u_{10}(t)f(t-10)\right],$$

which allows for an "easy" computation of the Laplace transform of g. Namely, we get

$$\begin{aligned}
\mathcal{L}(g) &= \frac{1}{5}\mathcal{L}\left(u_5(t)f(t-5)\right) - \frac{1}{5}\mathcal{L}\left(u_{10}(t)f(t-10)\right) \\
&= \frac{1}{5}e^{-5s}\mathcal{L}(f) - \frac{1}{5}e^{-10s}\mathcal{L}(f) \\
&= \frac{e^{-5s} - e^{-10s}}{5} \cdot \frac{1}{s^2}.
\end{aligned}$$

With the initial conditions, the left-hand side of the differential equation becomes

$$\mathcal{L}(y'' + 4y) = (s^2 + 4)Y(s),$$

where $Y(s) = \mathcal{L}(y)(s)$.

We therefore get (and define H)

$$Y(s) = \frac{e^{-5s} - e^{-10s}}{5} \cdot \frac{1}{s^2(s^2+4)} = \frac{e^{-5s} - e^{-10s}}{5} H(s).$$

Doing the same thing, we wish to find an $h(t) = \mathcal{L}^{-1}(H)(t)$ and then we can write

$$y(t) = \frac{1}{5}\left[u_5(t)h(t-5) - u_{10}(t)h(t-10)\right].$$

To compute the inverse Laplace transform of H we write

$$\frac{1}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4},$$

leading to the equation

$$1 = As(s^2+4) + B(s^2+4) + (Cs+D)s^2.$$

The right-hand side can be expanded to get

$$(A+C)s^3 + (B+D)s^2 + 4As + 4B,$$

meaning that $B = \frac{1}{4}$, $A = 0$ $C = 0$ and $D = -\frac{1}{4}$ and so

$$H(s) = \frac{1/4}{s^2} - \frac{1/4}{s^2+4} = \frac{1}{4} \cdot \frac{1}{s^2} - \frac{1}{8} \cdot \frac{2}{s^2+2^2},$$

and so the inverse Laplace transform is

$$h(t) = \frac{1}{8}\left[2t - \sin(2t)\right].$$

□

23. Lecture 23

We now move onto modeling something called the *Dirca delta function*, denoted $\delta(t)$, which is not technically a function. Let's start with a function

$$d_\tau(t) = \begin{cases} \frac{1}{2\tau} & |t| < \tau \\ 0 & |t| \geq \tau. \end{cases} .$$

This function d_τ is an actual function and

$$I(\tau) := \int_{-\tau}^{\tau} d_\tau(t)\, dt = 1$$

is defined for all $\tau > 0$. We also observe:

$$\lim_{\tau \to 0^+} d_\tau(t) = 0 \qquad \text{for all } t \neq 0,$$

and

$$\lim_{\tau \to 0^+} I(\tau) = 1.$$

This means that the "limit function" of d_τ, which we write as $\delta(t)$, should satisfy the following:

$$\delta(t) = 0, \quad \text{if } t \neq 0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(t)\, dt = 1.$$

Similarly, we write $\delta(t - t_0)$ for the function

$$\delta(t - t_0) = 0, \quad \text{if } t \neq t_0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(t - t_0)\, dt = 1.$$

What does the function $\delta(t)$ represent? It is supposed to represent (in this situation) a force that occurs at a single instance. For example, (and this does not apply directly to differential equations, but instead to partial differential equations) how sound waves propagate is governed by some physical differential equations. If you walk into a large building, such as a cathedral, with a balloon and pop it, the "force" term is approximated by a delta-function. That is because the balloon produces a very loud noise over a very short period of time.

Let's try to compute some things about the δ function. For example what is (for continuous f)

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)\, dt?$$

Well since $\delta = \lim_{\tau\to 0} d_\tau$, we expect

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)\delta(t-t_0)\,dt &= \lim_{\tau\to 0}\int_{-\infty}^{\infty} f(t)d_\tau(t-t_0)\,dt\\ &= \lim_{\tau\to 0}\frac{1}{2\tau}\int_{t_0-\tau}^{t_0+\tau} f(t)\,dt\end{aligned}$$

and for any $a < b$ there exists some t^* between a and b such that

$$\int_a^b f(t)\,dt = (b-a)f(t^*).$$

That means

$$\frac{1}{2\tau}\int_{t_0-\tau}^{t_0+\tau} f(t)\,dt = f(t^*_\tau),$$

where t^*_τ is between $t_0-\tau$ and $t_0+\tau$. And so as τ goes to 0, the value of t^*_τ converges to t_0.

Therefore

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)\,dt = \lim_{\tau\to 0} f(t^*_\tau) = f(t_0).$$

If $t_0 > 0$ then for τ small enough $d_\tau(t-t_0) = 0$ for all $t \le 0$, and so the above actually can be stated as

$$\int_0^{\infty} f(t)\delta(t-t_0)\,dt = f(t_0) \qquad \text{if } t_0 > 0.$$

Example 23.1. Compute the Laplace transform of $\delta(t-t_0)$, for $t_0 > 0$.

***Solution*:** Well we have

$$\mathcal{L}(\delta(t-t_0))(s) = \int_0^{\infty} e^{-st}\delta(t-t_0)\,dt.$$

This means that we can use the previous computations to show that

$$\int_0^{\infty} e^{-st}\delta(t-t_0)\,dt = e^{-st_0}.$$

That is exactly

$$e^{st_0} = s\mathcal{L}(u_{t_0}(t))(s) - u_{t_0}(0),$$

and the right-hand side is (formally) the Laplace transform of the derivative of $u_{t_0}(t)$. □

Example 23.2. Compute the solutions of

$$y'' + 2y' + y = \delta(t-10), \qquad y(0) = y'(0) = 0.$$

***Solution*:** We observe that based on the initial conditions we have

$$\mathcal{L}\left(y'' + 2y' + y\right)(s) = s^2Y(s) + 2sY(s) + Y(s) = (s+1)^2Y(s).$$

The right-hand side has Laplace transform

$$\mathcal{L}(\delta(t-10))(s) = e^{-10s}.$$

Thus

$$Y(s) = \frac{e^{-10s}}{(s+1)^2} = e^{-10s}H(s),$$

where $H(s) = (s+1)^{-2}$, which is already in it's partial fraction decomposition. The inverse Laplace transform of $H(s)$ is

$$\mathcal{L}(H)(t) = te^{-t},$$

and so

$$y(t) = u_{10}(t)(t-10)e^{-(t-10)}.$$

Note that te^{-t} solves the initial value problem

$$u'' + 2u' + u = 0 \qquad \text{and} \qquad u(0) = 0, \quad u'(0) = 1.$$

So the solution $y(t)$ satisfies:

$$\begin{cases} y'' + 2y' + y = 0, & y(0) = 0, \quad y'(0) = 0 \quad : 0 \le t < 10 \\ y'' + 2y' + y = 0, & y(10) = 0, \quad y'(10) = 1 \quad : 10 \le t \end{cases}.$$

So we can view this δ function as changing the velocity instantly. □

24. Lecture 24:

We state a theorem about Laplace transforms:

Theorem 24.1. *Suppose that $\mathcal{L}(f)(s) = F(s)$ and $\mathcal{L}(g)(s) = G(s)$ for all $s > a \geq 0$. Then $H(s) = F(s)G(s)$ is the Laplace transform of a funciton $h(t)$ defined by*

$$h(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau = (f * g)(t).$$

Here are some properties of the convolution

$$\begin{gathered} f * g = g * f \\ f * (ag + bh) = a(f * g) + b(f * h) \\ f * 0 = 0 * f = 0 \\ f * (g * h) = (f * g) * h. \end{gathered}$$

Example 24.1. Compute the inverse Laplace transform of

$$\frac{a}{s^2(s^2+a^2)}.$$

***Solution*:** We can either use partial fractions to solve this or we can use the convolution previously discussed. We set $F(s) = \frac{1}{s^2}$ and $G(s) = \frac{a}{s^2+a^2}$. We wish to find the inverse Laplace transform of

$$F(s)G(s) = \frac{a}{s^2(s^2+a^2)}.$$

It's easy to see that $\mathcal{L}^{-1}(F) = f(t) = t$ and $g(t) = \mathcal{L}^{-1}(G) = \sin(at)$. Thus

$$\mathcal{L}^{-1}(FG) = (f * g)(t).$$

We try to compute the convolution

$$\begin{aligned} f * g(t) &= \int_0^t (t-\tau)\sin(a\tau)\,d\tau \\ &= \int_0^t t\sin(a\tau) - \tau\sin(a\tau)\,d\tau \end{aligned}$$

$$\begin{aligned} \int_0^t \tau\sin(a\tau)\,d\tau &= \frac{-1}{a}\tau\cos(a\tau)\Big|_{\tau=0}^{\tau=t} + \frac{1}{a}\int_0^t \cos(a\tau)\,d\tau \\ &= -\frac{t\cos(at)}{a} + \frac{1}{a^2}\sin(at) \\ t\int_0^t \sin(a\tau)\,d\tau &= \frac{-t}{a}\cos(at) + \frac{t}{a}. \end{aligned}$$

Thus

$$\mathcal{L}^{-1}\left(\frac{a}{s^2(s^2+a^2)}\right)(t) = f * g(t) = \frac{at - \sin(at)}{a^2}.$$

□

Let's see the power of the convolution:

Example 24.2. Solve the initial value problem

$$y'' + 49y = g(t)$$

and

$$y'(0) = 3, \qquad y(0) = -1.$$

***Solution*:** We write the Laplace transform of g as $G(s) = \mathcal{L}(g)(s)$ and $Y(s) = \mathcal{L}(y)(s)$.

Then

$$\mathcal{L}(y'') = s^2 Y(s) + s - 3.$$

Therefore,

$$s^2 Y(s) + s - 3 + 49Y(s) = G(s),$$

and rewriting this gives

$$Y(s) = \frac{3-s}{s^2+49} + \frac{1}{s^2+49} G(s).$$

If we write $\Phi(s) = \frac{3-s}{s^2+4}$ and $H(s) = \frac{1}{s^2+4}$, then we can rewrite

$$Y(s) = \Phi(s) + H(s)G(s).$$

We can write

$$\Phi(s) = \frac{3}{7}\frac{7}{s^2+49} - \frac{s}{s^2+49}$$

That means

$$\phi(t) = \mathcal{L}^{-1}(\Phi)(t) = \frac{3}{7}\sin(7t) - \cos(7t).$$

Similarly,

$$h(t) = \mathcal{L}^{-1}(H)(t) = \frac{1}{7}\sin(7t).$$

Thus we can write

$$y(t) = \phi(t) + (h * g)(t) = \frac{3}{7}\sin(7t) - \cos(7t) + \frac{1}{7}\int_0^t \sin(7(t-\tau))g(\tau)\, d\tau.$$

□

This property works in general. Suppose that

$$ay'' + by' + cy = g(t), \qquad y(0) = y_0, \quad y'(0) = y_0'.$$

Then we can take the Laplace transforms of both sides to get

$$(as^2 + bs + c)Y(s) - (ay_0 s + by_0' + ay_0) = G(s).$$

We can re-write this as

$$Y(s) = \frac{ay_0 s + by_0' + ay_0}{as^2 + bs + c} + \frac{1}{as^2 + bs + c} \cdot G(s).$$

If

$$\Phi(s) = \frac{ay_0 s + by_0' + ay_0}{as^2 + bs + c}$$

and

$$H(s) = \frac{1}{as^2 + bs + c},$$

then

$$Y(s) = \Phi(s) + H(s)G(s).$$

Letting ϕ and h be the inverse Laplace transforms of Φ and H respectively allows us to write

$$y(t) = \phi(t) + (h * g)(t).$$

What do these functions represent?

The function $\phi(t)$ ends up solving the initial value problem

$$ay'' + by' + cy = 0 \qquad y(0) = y_0 \quad y'(0) = y_0'.$$

This term is where the initial conditions come into play.

The function h ends up solving this initial value problem

$$ay'' + by' + cy = \delta(t) \qquad y(0) = y'(0) = 0.$$