Week 1

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Why study systems of differential equations?

- They arise naturally involving systems of several dependent variables, i.e. motion in a magnetic field.
- They can be used to reduce higher order ODEs into first order ODEs.

Let's examine the second situation.

<u>Ex</u>) The general spring mass system is

$$m u'' + \gamma u' + k u = f(t)$$

m = mass, γ = damping coefficient, k = spring constant, f = forcing function.

Consider the transform:

$$x_1 = u, \quad x_2 = u'.$$

Then

$$x_1' = u' = x_2$$

$$x_2' = u'' = \frac{1}{m}(f(t) - ku - \gamma u')$$

$$= \frac{1}{m} f(t) - \frac{k}{m} x_1 - \frac{\gamma}{m} x_2.$$

In matrix form we have

$$\vec{x}' = A\vec{x} + \vec{F}(t)$$

where

where $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\gamma}{m} \end{pmatrix}$,

$$\vec{F}(t) = \begin{pmatrix} 0 \\ \frac{1}{m} f(t) \end{pmatrix}.$$

$\blacksquare$

In general, an n^{th} order ODE looks like

$$y^{(n)} = f(t, y, y', y'', \dots, y^{(n-1)})$$

where $y^{(k)} = k^{th}$ derivative of y.

We can turn this into a system by setting

$$x_1 = y, \quad x_2 = y', \dots, x_n = y^{(n-1)}.$$

Then $x_1' = x_2$, $x_2' = x_3$, $\dots$,

$$x_n' = f(t, x_1, \dots, x_n).$$

A general linear system is of the form

$$\left.\begin{aligned} x_1 &= f_1(t, x_1, \dots, x_n) \\ x_2 &= f_2(t, x_1, \dots, x_n) \\ &\ \vdots \\ x_n &= f_n(t, x_1, \dots, x_n). \end{aligned}\right\} \quad (*)$$

<u>Rmk</u> I'll call the variable t time and the x_j's spacial variables.

<u>Def</u> The system $(*)$ is said to be <u>linear</u> if $\forall j$ (for all j)

$$f_j(t, x_1, \dots, x_n) = \sum_{k=1}^{n} P_{j,k}(t)\, x_k + g_j(t)$$

where $P_{j,k}$ and g_j are functions only of time. Systems which are not linear are called <u>nonlinear</u>.

<u>Def</u> A linear system is said to be <u>homogeneous</u> if $g_j \equiv 0, \ \forall j$.

Def) A linear system is said to be homogeneous if $g_j \equiv 0, \forall j$.

Systems which are not homogeneous are said to be nonhomogeneous.

Remark) A linear system can be written as

$$\vec{x}' = P(t)\vec{x} + \vec{g}(t)$$

where $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $\vec{g} = \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix}$, $\tilde{P} = \begin{pmatrix} P_{1,1}, & P_{1,2} & \cdots & P_{1,n} \\ P_{2,1} & P_{2,2} & \cdots & P_{2,n} \\ \vdots & & \ddots & \ddots \\ P_{n,1} & P_{n,1} & & P_{n,n} \end{pmatrix}$.

Review of Matrices.

Remark on notation:) I'll try to use capital letters for matrices, and denote vectors with a $\rightarrow$ over the letters. I will fail at always doing this.

Def) An $m \times n$ matrix $A = (a_{i,j})$ where $i \le m, j \le n$ is written as

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & & & \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

m rows
n columns.

Def) The transpose of an $m \times n$ matrix $A = (a_{ij})$ is the $n \times m$ matrix denoted by A^t by

$$A^t = \begin{pmatrix} a_{1,1} & a_{2,1} & \cdots & a_{m,1} \\ a_{1,2} & a_{2,2} & & \vdots \\ \vdots & \vdots & & \\ a_{1,n} & a_{2,n} & & a_{m,n} \end{pmatrix}.$$

If a_{ij} are complex, we denote $\bar{A} = (\overline{a_{ij}})$ and call this the conjugate of A.

and call this the <u>conjugate</u> of A.

The <u>adjoint</u> of A, denoted $A^* := (\overline{A})^t = \overline{(A^t)}$.

<u>Eg</u>) $A = \begin{pmatrix} 3 & 2-i \\ 4+i & -5+2i \end{pmatrix}$ then

$A^t = \begin{pmatrix} 3 & 4+i \\ 2-i & -5+2i \end{pmatrix}$, $\overline{A} = \begin{pmatrix} 3 & 4-i \\ 2+i & -5-2i \end{pmatrix}$

$A^* = \begin{pmatrix} 3 & 2+i \\ 4-i & -5-2i \end{pmatrix}$.

<u>Properties</u>) • $A = (a_{ij})$ is <u>equal</u> to $B = (b_{ij})$ if $a_{ij} = b_{ij}$ $\forall i, j$.

- $A + B = (a_{ij} + b_{ij})$
- $A + B = B + A$, $(A+B) + C = A + (B + C)$.
- $\lambda A = (\lambda a_{ij})$, $\lambda(A+B) = \lambda A + \lambda B$, $(\lambda + \mu) A = \lambda A + \mu A$.
- If A is $m \times n$ and B is $n \times k$ then $C := AB$ is the $m \times k$ matrix whose elements are c_{ij} defined by

$$c_{ij} = \sum_{l=1}^{n} a_{i,l}\, b_{l,j} .$$

- If dimensions match
  $$A(B+C) = AB + AC$$
  $$(AB)C = A(BC)$$
  but in general $AB \neq BA$.

- If $\vec{x}, \vec{y}$ are vectors then
  $$\vec{x}^t \vec{y} = \sum_{1}^{n} x_j y_j$$
  and we define the <u>inner product</u>
  $$(\vec{x}, \vec{y}) = \vec{x}^t \bar{\vec{y}} = \sum_{1}^{n} x_j \bar{y}_j .$$

- The magnitude of a vector $\vec{x}$ is
  $$(\vec{x}, \vec{x})^{1/2} = \sqrt{\sum_{1}^{n} |x_j|^2} .$$

- The $n \times n$ <u>identity matrix</u> is the matrix
  $$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ 0 & 0 & 1 & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0\ 0 & \cdots\ 0 & 1 \end{pmatrix}$$

- An $n \times n$ matrix A is said to be <u>invertible</u> if there exists a matrix B such that $AB = BA = I$. Denote $A^{-1} = B$.
  We call a square matrix which is not invertible <u>singular</u>.

Computing the inverse

- Cramer's Rule:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

1) Define M_{ij} as the determinant of the $(n-1)\times(n-1)$ matrix obtained by deleting the i^{th} row & j^{th} column from A.

<u>Eg</u>)
$$A = \begin{pmatrix} 4 & 2 & 3 & 1 \\ 1 & 1 & 7 & 0 \\ 2 & 4 & 8 & 0 \\ 3 & 2 & 4 & 0 \end{pmatrix}$$

$$M_{2,3} = \det\begin{pmatrix} 4 & 2 & 1 \\ 2 & 4 & 0 \\ 3 & 2 & 0 \end{pmatrix} = 4 - 12 = -8.$$

2) Define
$$C_{ij} = (-1)^{i+j} M_{ij}$$

3) The inverse is $B = (b_{ij})$ where
$$b_{ij} = \frac{C_{ji}}{\det(A)}.$$

- Row reduction / Gaussian elimination:

1) Form the augmented matrix $A \mid I$

2) Interchange rows, multiply rows by nonzero scalars and add multiples of rows to other rows to turn the left-half of $A \mid I$ to I.
When this is done the RHS is A^{-1}.

Eg)
$$A = \begin{pmatrix} 1 & -1 & -1 \\ 3 & -1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$$

$$A \mid I = \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right)$$

add multiples of row 1 to the other rows to get

$$\left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 5 & -3 & 1 & 0 \\ 0 & 4 & 5 & -2 & 0 & 1 \end{array}\right)$$

Divide row 2 by 2.

$$\left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 5/2 & -3/2 & 1/2 & 0 \\ 0 & 4 & 5 & -2 & 0 & 1 \end{array}\right)$$

Subtract 4·row 2 from row 3, and add row 2 to row 1.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3/2 & -1/2 & 1/2 & 0 \\ 0 & 1 & 5/2 & -3/2 & 1/2 & 0 \\ 0 & 0 & -5 & 4 & -2 & 1 \end{array}\right)$$

Divide row 3 by −5.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3/2 & -1/2 & 1/2 & 0 \\ 0 & 1 & 5/2 & -3/2 & 1/2 & 0 \\ 0 & 0 & 1 & -4/5 & 2/5 & -1/5 \end{array}\right)$$

Subtract multiples of row 3 from the others:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/10 & -1/10 & 3/10 \\ 0 & 1 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & -4/5 & 2/5 & -1/5 \end{array}\right) = I \mid A^{-1}.$$

So
$$A^{-1} = \begin{pmatrix} 7/10 & -1/10 & 3/10 \\ 1/2 & -1/2 & 1/2 \\ -4/5 & 2/5 & -1/5 \end{pmatrix}.$$

Matrix functions:

We write

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} a_{ij}(t) \end{pmatrix}.$$

Def) A is said to be <u>continuous</u> (resp <u>differentiable</u>) if a_{ij} is continuous (resp. differentiable).

We define the <u>derivative</u> of A, denoted by $A'(t)$ or $\frac{d}{dt}A(t)$ by

$$A'(t) = \begin{pmatrix} a_{ij}'(t) \end{pmatrix}.$$

and the <u>integral</u> from a to b of A by

$$\int_a^b A(t)\,dt = \left(\int_a^b a_{ij}(t)\,dt \right)_{ij}$$

Eg) $A(t) = \begin{pmatrix} \sin t & 3 \\ \cos t & e^{2t} \end{pmatrix}$

$$A'(t) = \begin{pmatrix} \cos t & 0 \\ -\sin t & 2e^{2t} \end{pmatrix}$$

$$\int_0^t A(s)\,ds = \begin{pmatrix} -\cos(t) + 1 & 3t \\ \sin t & \frac{1}{2}(e^{2t} - 1) \end{pmatrix}.$$

<u>Properties)</u> If A and B are of the proper dimensions and C is a constant matrix then

- $(CA)' = CA'$
- $(A+B)' = A' + B'$
- $(AB)' = AB' + A'B$

Eigen stuff:

Recall:

Def If A is an $n \times n$ matrix, we say that λ is an eigenvalue if
$$\det(A - \lambda I) = 0.$$
If λ is an eigenvalue, then we say $\vec{v}$ is an eigenvector (with eigenvalue λ) then
$$A\vec{v} = \lambda\vec{v}.$$
The characteristic equation is the n^{th} degree polynomial
$$\det(A - \lambda I).$$

Thm (Fundamental theorem of algebra) If $p_n(x)$ is an n^{th} degree polynomial w/ complex coefficients then there exists n complex roots counted w/ multiplicity. I.e.
$$p_n(x) = c(x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n).$$

Cor An $n \times n$ matrix A has n eigenvalues $\lambda_1, \dots, \lambda_n$ counted with multiplicity.

Def If the characteristic equation for A has λ as a root which appears m times in the expansion above then we say λ has algebraic multiplicity of m.

The number of linearly independent eigenvectors with eigenvalue λ is called the geometric multiplicity of λ.

Property geometric mult of λ $\leq$ algebraic mult of λ.

Examples:

Examples:

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -\lambda^3 + 3\lambda + 2 \Rightarrow$$

eigenvalues are

$$\lambda_1 = 2, \quad \lambda_2 = \lambda_3 = -1.$$

What is the eigenvector w/ eigenvalue 2?

$$(A - 2I)\vec{x} = 0 \iff \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

If $x_1 = x_2 = x_3$ then we solve this

thus the eigenvector is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

For $\lambda = -1$ we have

$$(A + I)\vec{x} = 0 \iff \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

let's row reduce

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \text{ to } \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So $x_1 + x_2 + x_3 = 0$

this can be solved by

$$x_1 = c_1, \quad x_2 = c_2, \quad x_3 = -c_1 - c_2$$

So the eigenvectors can be

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \cdot \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

The above example hits at an important point:

<u>Thm</u> If $A^* = A$ then the following hold:

1) All eigenvalues are real
2) geom and alg. multiplicities are the same.
3) If $\vec{v}$ and $\vec{w}$ are eigenvectors w/ diff eigenvalues then $(\vec{v}, \vec{w}) = 0$.
4) If λ has alg. mult. $m \geq 1$, then there are m orthogonal eigenvectors w/ eigenvalue λ.