Week 2

Sunday, June 30, 2019 9:26 AM

Complex eigenvalues

What happens when the eigenvalues of a matrix are complex?

Ex.

Consider the 2x2 matrix

$$A = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix}.$$

We compute the characteristic equation as

$$\det(A - \lambda I) = \left(-\frac{1}{2} - \lambda\right)^2 + 1$$
$$= \lambda^2 + \lambda + 5/4.$$

This has roots

$$\lambda = -\frac{1}{2} \pm i.$$

What are the eigenvectors?

Let's find the eigenvector for $\lambda_1 = -\frac{1}{2} - i$.

Well

$$A - \lambda I = \begin{pmatrix} i & 1 \end{pmatrix}$$

well

$$A - \lambda I = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}$$

and we need

$$\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \vec{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We have the two equation

$$i v_1 + v_2 = 0$$

$$-1 v_1 + i v_2 = 0$$

Multiply the first by i to get

$$-v_1 + i v_2 = 0$$

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so $\vec{v} = \begin{pmatrix} i \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ -i \end{pmatrix}$

since one is a constant mult. of the other.

Similarly the eigenvector w/ eigenvalue $\lambda_2 = \frac{-1}{2} + i$ is

$$\begin{pmatrix} 1 \\ i \end{pmatrix}.$$

How do we change basis ?

How do we change basis here?

Write $\vec{v}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ and

$\vec{v}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$.

Also write $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

as the basis for $\mathbb{C}^2$.

We note

$$\vec{v}_1 = \vec{e}_1 - i\,\vec{e}_2$$
$$\vec{v}_2 = \vec{e}_1 + i\,\vec{e}_2 .$$

Thus the change of basis from $\{\vec{v}_1, \vec{v}_2\}$ to $\{\vec{e}_1, \vec{e}_2\}$ is $P = \begin{bmatrix} 1 & 1 \\ -i & +i \end{bmatrix}$

Going from $\{\vec{e}_1, \vec{e}_2\}$ to $\{\vec{v}_1, \vec{v}_2\}$ is $P^{-1} = \frac{1}{2i}\begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2}i \\ \frac{1}{2} & -\frac{1}{2}i \end{bmatrix}$

Thus we have

$$(\mathbb{C}^2, \mathcal{V}) \xrightarrow{D} (\mathbb{C}^2, \mathcal{V})$$

$$\begin{array}{ccc} (\mathbb{C}^2, \mathcal{V}) & \xrightarrow{D} & (\mathbb{C}^2, \mathcal{V}) \\ P \downarrow & & \uparrow P^{-1} \\ (\mathbb{C}^2, \mathcal{E}) & \xrightarrow{A} & (\mathbb{C}^2, \mathcal{E}) \end{array} \quad (\otimes)$$

So $D = P^{-1} A P = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

What does $\otimes$ mean?

$\mathcal{V}$ is the basis $\{\vec{v}_1, \vec{v}_2\}$

$\mathcal{E}$ is the basis $\{\vec{e}_1, \vec{e}_2\}$.

To go from basis $\mathcal{V}$ to $\mathcal{E}$ we multiply by P.

To then apply the transformation we multiply by A, but we are still in basis $\mathcal{E}$. We go back to $\mathcal{V}$ by multiplying by $\mathcal{V}$.

What about this example holds in general?

<u>Prop</u> Suppose A is an

Prop/ Suppose A is an $n \times n$ matrix with real entries. If $\lambda \in \mathbb{C}$, is a non-real eigenvalue of A then

- $\bar{\lambda}$ is an eigenvalue
- If $\vec{v}$ is λ's eigenvector then $\overline{\vec{v}}$ is the eigenvector for $\bar{\lambda}$.

Prop/ Suppose A is $n \times n$ with complex entries. If $\vec{v}_1, \ldots, \vec{v}_n$ are n linearly independent eigenvectors with eigenvalues $\lambda_1, \ldots, \lambda_n$ respectively then

$$A = PDP^{-1}$$

for $P = [\vec{v}_1, \ldots, \vec{v}_n]$

and $D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$

where D is diagonal.

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This brings us to an important concept: Diagonalization.

Def: A matrix, A, is <u>diagonalizable</u> if $\exists$ an invertible matrix P and a diagonal matrix D s.t. $A = PDP^{-1}$.

What happens here? Multiply both sides by P on the right to get

$$AP = PD,$$

write

$$P = [\vec{v}_1, \ldots \vec{v}_n].$$

$$D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

Then

$$AP = [A\vec{v}_1, \ldots A\vec{v}_n]$$

$$PD = [\lambda_1 \vec{v}_1 \ldots \lambda_n \vec{v}_n]$$

and so

$$A\vec{v}_k = \lambda_k \vec{v}_k.$$

Thus P's columns are eigenvectors.

As we have seen a Matrix A is diagonalizable if and only if A has a basis of eigen vectors. (i.e. there are n linearly independent eigenvectors).

We've also seen a 2×2 example where there is only 1 eigenvector: $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

The previous 2×2 example is an example of a Jordan block.

<u>Def</u> A <u>Jordan block</u> of size m and eigenvalue λ

size m and eigenvalue λ
is the $m \times m$ matrix:

$$J = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 \cdots & 0 \\ \vdots & & \ddots & \ddots & 1 & 0 \\ & & & \ddots & \lambda & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & \lambda \end{pmatrix}.$$

There are λ's on the diagonal and 1's on the superdiagonal.

Eg]

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} \pi & 1 \\ 0 & \pi \end{pmatrix}$$

Def] A matrix B is in <u>Jordan normal form</u> if

$$B = \begin{pmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_k \end{pmatrix}.$$

where J_ℓ are $m_\ell \times m_\ell$ Jordan blocks.

Eg]

$$\begin{pmatrix} 3 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 4 & 0 & 0 \end{pmatrix}$$

Eg/

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} \pi & 1 & 0 & 0 \\ 0 & \pi & 1 & 0 \\ 0 & 0 & \pi & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Thm/ If A is an $n \times n$ matrix then there exists an invertible P, and an $n \times n$ matrix B s.t. $A = PBP^{-1}$, $B = P^{-1}AP$, such that B is in Jordan normal form. B is unique up to reordering.

Prop/ If J is a Jordan block of size m and eigenvalue λ then

- λ is the only eigenvalue.
- λ has alg. mult. m
- λ has geometric multiplicity 1.

Prop/ If B is in Jordan

normal form with blocks $J_1, J_2, \ldots, J_k$, where J_ℓ is of size m_ℓ and eigenvalue λ_ℓ then

- $\lambda_1, \ldots, \lambda_k$ are the only eigenvalues.
- If $\lambda = \lambda_{i_1} = \lambda_{i_2} = \cdots = \lambda_{i_p}$ (ie λ is the eigenvalue for p separate Jordan blocks)

then

1) λ has alg. mult. $\sum_{j=1}^{p} m_{i_j}$

2) λ has geom. mult. p.

Eg) $\begin{pmatrix} \pi & 1 & 0 & 0 \\ 0 & \pi & 0 & 0 \\ 0 & 0 & \pi & 1 \\ 0 & 0 & 0 & \pi \end{pmatrix}$ has eigenvalue π, but π has geometric mult. 2.

<u>Generalized eigenvectors</u>: $\vec{v}$ is a generalized eigenvector of rank m and eigenvalue λ if

$$(A - \lambda I)^m \vec{v} = \vec{0},$$

$$(A - \lambda I)^{m-1} \vec{v} \neq \vec{0}.$$

$$(A - \lambda I)^{m-1} \vec{v} \neq \vec{0}.$$

Rmk) If J is a Jordan block of size m and eigenvalue λ and $\vec{e}_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j^{th}$ spot

then $\vec{e}_j$ is a generalized eigenvalue of rank j.

$\vec{e}_j$ are special in the above example because they for an example of a Jordan chain.

Def) A Jordan chain with eigenvalue λ for A is a sequence of generalized eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ such that

$$A\vec{v}_k = \lambda \vec{v}_k + \vec{v}_{k-1} \quad \text{for } k = 2, 3, \ldots, m.$$

Thm) Every square matrix A has a basis of generalized eigenvectors a basis of generalized eigenvectors and that basis can be taken to be a sequence of Jordan chains. ☑

<u>Takeaway</u> An $n \times n$ matrix A may not have n linearly independent eigenvectors, but it has something close. There are n vectors

$$\vec{v}_1^{(1)}, \ldots, \vec{v}_{m_1}^{(1)}, \vec{v}_1^{(2)}, \ldots, \vec{v}_{m_2}^{(2)}, \ldots, \vec{v}_1^{(k)}, \ldots, \vec{v}_{m_k}^{(k)}$$

which are linearly independent and $\vec{v}_1^{(\ell)}, \ldots, \vec{v}_{m_\ell}^{(\ell)}$ is a Jordan chain of size m_ℓ with some eigenvalue λ_ℓ.

With this collection of vectors we can write

$$P = \begin{bmatrix} | & | & & | & & | & & | \\ \vec{v}_1^{(1)} & \vec{v}_2^{(1)} & \cdots & \vec{v}_{m_1}^{(1)} & \cdots & v_1^{(k)} & \cdots & v_{m_k}^{(k)} \\ | & | & & | & & | & & | \end{bmatrix}$$

ie the rows are the Jordan chains

Then

Then

$$B = P^{-1} A P \quad \text{is the}$$

Jordan normal form of A.

Eg) Consider $A = \frac{1}{2}\begin{pmatrix} 3 & 1 & -1 \\ -2 & 6 & 6 \\ 1 & -1 & 3 \end{pmatrix}$

and $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

Note

$$A\vec{v}_1 = \frac{1}{2}\begin{pmatrix} 3 & +1 \\ -2 & +6 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = 2\vec{v}_1$$

$$A\vec{v}_2 = \frac{1}{2}\begin{pmatrix} 1+1 \\ 6+0 \\ -1+-3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = 2\vec{v}_2 + \vec{v}_1$$

$$A\vec{v}_3 = \frac{1}{2}\begin{pmatrix} 3+2-1 \\ -2+12+0 \\ 1-2+3 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} = 2\vec{v}_3 + \vec{v}_1$$

So the vectors

$\vec{v}_1, \vec{v}_2, \vec{v}_3$ form a Jordan chain

and if

$$P = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{pmatrix}$$

then

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} = P^{-1} A P$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} = P^{-1} A P$$

Differential equations.

Consider a system

$$\vec{x}' = A\vec{x}, \qquad \vec{x}(0) = \vec{x}_0$$

What happens if

$\vec{x}(t) = f(t) \cdot \vec{v}$ for some function f and a vector $\vec{v}$?

Well

$$\vec{x}'(t) = \frac{d}{dt}\left(f(t)\, \vec{v} \right) = f'(t)\, \vec{v}.$$

But
$$\begin{aligned} \vec{x}'(t) &= A\, \vec{x}(t) \\ &= A\, f(t)\, \vec{v} \\ &= f(t)\, A\vec{v}. \end{aligned}$$

Thus if

$\vec{x}(t) = f(t)\vec{v}$ and $\vec{x}' = A\vec{x}$ then
$$f'(t)\, \vec{v} = f(t)\, A\vec{v}.$$

What happens if $\vec{v}$ is an

What happens if $\vec{v}$ is an eigenvector?

Well then

$$f'(t)\,\vec{v} = f(t)\,A\,\vec{v}$$
$$= \lambda f(t)\,\vec{v}.$$

So $f' = \lambda f \Rightarrow f(t) = ce^{\lambda t}$.

Eg] Consider

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad \vec{X}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Then $x_1'(t) = 3x_1(t)$, $x_1(0) = 2$

$x_2'(t) = 2x_2(t)$, $x_2(0) = 3$

Thus $x_1(t) = 2e^{3t}$, $x_2(t) = 3e^{2t}$.

and $\vec{X}(t) = \begin{pmatrix} 2e^{3t} \\ 3e^{2t} \end{pmatrix}$.

The importance of the above example is that $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are eigenvectors.

Thm] Suppose A has n linearly ind. eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$, and eigenvalues $\lambda_1, \ldots, \lambda_n$.

Let $\vec{x}'(t) = A\vec{x}(t)$ and $\vec{X}(0) = \vec{x}_0$.

Then

$$X(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 + \cdots + c_n e^{\lambda_n t} \vec{v}_n$$

is the solution where

$$\vec{X}_0 = c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n \quad \text{is}$$

the unique way of writing $\vec{x}$ in the basis $(\vec{v}_1, \ldots, \vec{v}_n)$.

Rmk/ This works for both real and complex matrices A, and real and complex eigenvectors/values.

However we often want real solutions when A has real entries.

For example, if

$$A = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix},$$

which has eigenvalues

$$\vec{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}, \vec{v}_2 \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

and $\lambda_1 = -\frac{1}{2} + i$, $\lambda_2 = -\frac{1}{2} - i$. (computed earlier).

The solution to

$$\vec{x}'(t) = A\vec{x}(t). \quad \vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\vec{x}\,'(t) = A\vec{x}(t), \qquad \vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

can be given by

$$\otimes \quad \vec{x}(t) = \frac{1}{2} e^{(-\frac{1}{2}+i)t}\begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{1}{2} e^{(-\frac{1}{2}-i)t}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

is not satisfying.

We note

$$\vec{x}(t) = \frac{1}{2}\begin{pmatrix} e^{it} + e^{-it} \\ i e^{it} - i e^{-it} \end{pmatrix} e^{-t/2}$$

$$= \frac{1}{2}\begin{pmatrix} \cos(t) + i\sin(t) + \cos(t) - i\sin(t) \\ i(\cos(t) + i\sin(t)) - i(\cos(t) - i\sin t) \end{pmatrix} e^{-t/2}$$

$$= \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} e^{-t/2}$$ is a better way to write this.