Week 3 Matrix Exponential

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To make notation clearer I will write

$$\exp(z) = e^{z}$$

where z could be anything.

Recall Math 307 (or 125)

$$y' = ay \qquad \text{where } a \in \mathbb{R}$$

implies that $y(t) = y(0)\, e^{at}$.

Indeed recall that

$$\exp(at) = 1 + at + \frac{1}{2!}(at)^2 + \frac{1}{3!}(at)^3 + \cdots$$

$$\frac{d}{dt}\exp(at) = 0 + a + a^2 t + \frac{a}{2!}(at)^2 + \cdots$$

$$= a\exp(at)$$

and since $y(0)$ is a constant we get

$$\frac{d}{dt}\left(\exp(at)\, y(0)\right) = a\exp(at)\, y(0).$$

Now consider (what we've been doing recently)

$$\vec{x}\,' = A\vec{x}. \qquad \vec{x}(0) = \vec{x}_0$$

Is it possible
$$\vec{x}(t) = \exp(At)\,\vec{x}_0\ ?$$

Well what is $\exp(At)$?
It should be a generalization of $e^{at} = 1 + at + \frac{1}{2}(at)^2 + \cdots$
and be
$$\exp(At) = I + At + \frac{1}{2}(At)^2 + \frac{1}{3!}(At)^3 + \cdots$$
Let's differentiate:
$$\frac{d}{dt}\exp(At) = 0 + A + A^2 t + \frac{1}{2}A^3 t^2 + \cdots$$
$$= A\left(I + At + \frac{1}{2}(At)^2 + \frac{1}{3!}(At)^3 + \cdots\right)$$
$$= A\exp(At)$$

Since $\vec{x}(0) = \vec{x}_0$ is a constant
we should have
$$\frac{d}{dt}\left(\exp(At)\,\vec{x}_0\right) = A\exp(At)\,\vec{x}_0$$
$$= A\,\vec{x}(t).$$
Thus the solution is
$$\vec{x}(t) = e^{At}\,\vec{x}_0.$$

But we know
$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + \cdots + c_n e^{\lambda_n t}\vec{v}_n$$
when A is diagnolizable.
Is $\exp(At)\,\vec{x}_0 = c_1 e^{\lambda_1 t}\vec{v}_1 + \cdots + c_n e^{\lambda_n t}\vec{v}_n$?

Yes, note

$$A = PDP^{-1} \quad \text{where}$$

$$P = [\vec{v}_1 \cdots \vec{v}_n] \quad \text{and}$$

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

so

$$e^{At} = e^{PDP^{-1}t} = I + PDP^{-1}t + \tfrac{1}{2}(PD^2P^{-1})t^2 \cdots$$

$$= P\left(I + Dt + \tfrac{1}{2}(Dt)^2 + \cdots\right)P^{-1}$$

$$= P e^{Dt} P^{-1}$$

and $e^{Dt} = \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix}$

so

$$e^{At}\vec{x}_0 = Pe^{Dt}P^{-1}\vec{x}_0$$

$$= P\begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix}\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

$$= P\begin{pmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{pmatrix}$$

$$= c_1 e^{\lambda_1 t}\vec{v}_1 + \cdots + c_n e^{\lambda_n t}\vec{v}_n .$$

which is the same answer.

What about repeated eigenvalues?

let's try

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A^3 = 0 \cdots$$

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So

$$e^{At} = I + At + \underbrace{\tfrac{1}{2}A^2t^2 + \cdots}_{0}$$

$$= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{0t} & te^{0t} \\ 0 & e^{0t} \end{pmatrix}$$

In general (for 2×2)

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \Rightarrow e^{At} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$$

$$A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \Rightarrow e^{At} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} & \frac{1}{2}t^2e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{pmatrix}$$

What is special about matrix exponentials?

- Its an "easy" way to write down a solution, but its downside is its hard to compute.

Q) Can we use the matrix exponential to compute the solution to

$$\vec{x}\,' = A(t)\,\vec{x} \quad ?$$

After all, in 1 variable

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$$y' = a(t)\,y \quad \text{is solved}$$

by

$$y(t) = \exp\left(\int_0^t a(s)\,ds\right) y(0).$$

Answer) In general, no.
The problem is the chain rule for matrix exponentials.
Note for functions

f and g

$$fg = gf$$

but for some matrices A and B

$$AB \neq BA.$$

Thus $\frac{d}{dt} A^2(t)$ does not always equal

$$2\,A(t)\,A'(t).$$

Eg) $A = \begin{pmatrix} t^2 & t \\ 0 & 0 \end{pmatrix}$

$$A^2 = \begin{pmatrix} t^4 & t^3 \\ 0 & 0 \end{pmatrix} \quad \text{and}$$

$$A' = \begin{pmatrix} 2t & 1 \\ 0 & 0 \end{pmatrix}, \quad (A^2)' = \begin{pmatrix} 4t^3 & 3t^2 \\ 0 & 0 \end{pmatrix}$$

$$2AA' = 2\begin{pmatrix} t^2 & t \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2t & 1 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 4t^3 & 2t^2 \\ 0 & 0 \end{pmatrix} = (A^2)'$$

But $(A^2)' = A'A + AA'$

But $(A^2)' \neq A'A + AA'$ in general.

Since $\exp\left(\int_0^t A(s)\,ds\right)$

$$= I + \int_0^t A(s)\,ds + \frac{1}{2}\left(\int_0^t A(s)\,ds\right)^2 + \cdots$$

we <u>can't</u> get

$$\frac{d}{dt}\exp\left(\int_0^t A(s)ds\right) = A(t)\exp\left(\int_0^t A(s)ds\right).$$

We <u>can</u> if $\int_0^t A(s)ds\, A(t) = A(t)\int_0^t A(s)ds$

but not in general.

This can also be see with the problem

<u>Prop</u> Suppose A and B commute (i.e $AB = BA$) then $e^{A+B} = e^A e^B$.

In general $e^{A+B} \neq e^A e^B$.

What matrices commute?

Well if J is an $m\times m$ Jordan block w/ eigenvalue λ then

$$J = \Lambda + N$$

where $\Lambda = \operatorname{diag}(\lambda, \dots, \lambda)$

and $N = \begin{pmatrix} 0 & 1 & & 0 \\ 0 & 0 & 1 & \\ \vdots & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$

$$\begin{pmatrix} \vdots & \ddots & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$$

Note $\Lambda = \lambda I_{m\times m}$ and so

$\Lambda N = N \Lambda$.

Thus $e^{(\Lambda+N)t} = e^{\Lambda t} e^{Nt}$

What is $e^{\Lambda t}$?

well

$$e^{\Lambda t} = \begin{pmatrix} e^{\lambda t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda t} \end{pmatrix} = e^{\lambda t} I_{m\times m}$$

$$e^{Nt} = I + Nt + \frac{1}{2}(Nt)^2 + \cdots$$

note $N^2 = \begin{pmatrix} 0 & 0 & 1 & & \\ 0 & 0 & 0 & 1 & \\ 0 & \vdots & & \ddots & 1 \\ \vdots & & & & 0 \end{pmatrix}$

$$N^3 = \begin{pmatrix} 0 & 0 & 0 & 1 & \ddots & 0 \\ 0 & & 0 & 0 & \ddots & 1 \\ & & & & & 0 \end{pmatrix}$$

$$N^{m-1} = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ & & & 0 \\ & 0 & & \vdots \\ & & & 0 \end{pmatrix}, \quad N^m = 0$$

Thus

$$e^{Nt} = \begin{pmatrix} 1 & t & \frac{1}{2}t^2 & \cdots & \frac{1}{(m-1)!}t^{m-1} \\ 0 & 1 & t & & \frac{1}{(m-2)!}t^{m-2} \\ \vdots & 0 & 1 & & \vdots \\ \vdots & \vdots & \vdots & 1 & t \\ 0 & 0 & 0 & \cdots\ 0\ 0 & 1 \end{pmatrix}$$

So $e^{(\Lambda+N)t} = e^{Jt} = e^{\lambda t} e^{Nt}$

Moreover for Jordan blocks

$$B = \begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_\cdot \end{pmatrix}$$

$$\begin{pmatrix} & \ddots & \\ O & & J_k \end{pmatrix}$$

$$B^d = \begin{pmatrix} J_1^d & O & \cdots & O \\ & J_2^d & & O \\ O & & \ddots & J_k^d \end{pmatrix}$$ is still block diagonal

$$\exp(Bt) = \begin{pmatrix} \exp(J_1 t) & \ddots & O \\ O & & \exp(J_k t) \end{pmatrix}.$$

What about

$$\vec{x}' = A\vec{x} + \vec{b}(t) \quad \otimes$$

where $\vec{b}$ is some vector function.

If $A = PDP^{-1}$ we can rewrite $\otimes$ as

$$PP^{-1}\vec{x}' = PDP^{-1}\vec{x} + \vec{b}(t)$$

If we call $\vec{y} = P^{-1}\vec{x}$ then

$$P\vec{y}' = PD\vec{y} + \vec{b}(t)$$

multiplying by P^{-1} on the left we get

$$P^{-1}P\vec{y}' = P^{-1}PD\vec{y} + P^{-1}\vec{b}(t)$$

$$\vec{y}' = D\vec{y} + \underbrace{P^{-1}\vec{b}(t)}_{\vec{h}(t)}$$

So

$$\vec{y}' = D\vec{y} + \vec{h}(t)$$

$$\begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{pmatrix} = \begin{pmatrix} \lambda_1 y_1 \\ \lambda_2 y_2 \\ \vdots \\ \lambda_n y_n \end{pmatrix} + \begin{pmatrix} h_1(t) \\ h_2(t) \\ \vdots \\ h_n(t) \end{pmatrix}$$

So $\quad y_j' = \lambda_j y_j + h_j(t)$

which is solved by

$$y_j(t) = e^{\lambda_j t} \int_0^t e^{-\lambda_j s} h_j(s)\,ds + C_j e^{\lambda_j t}$$

and

$$\vec{x} = P\vec{y}(t).$$

Eg)

$$A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \quad \text{and}$$

$$\vec{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \vec{x} + \underbrace{\begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}}_{\vec{b}(t)}$$

Note A has the eigenvectors & eigenvalues

$$-3 \text{ w/ } \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad -1 \text{ w/ } \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Note

$$P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

has inverse

$$P^{-1} = \begin{pmatrix} 1 & -1 \end{pmatrix}$$

has inverse

$$P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

and so

$\vec{y} = P^{-1}\vec{x}$ solves

$$\vec{y}' = \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix}\vec{y} + \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$$

$$y_1' = -3\vec{y}_1 + e^{-t} - \frac{3}{2}t$$

$$y_2' = -\vec{y}_1 + e^{-t} + \frac{3}{2}t$$

$$y_1 = e^{-3t}\int_0^t e^{3s}\left(e^{-s} - \frac{3}{2}s\right)ds + c_1 e^{-3t}$$

$$= c_1 e^{-3t} + e^{-3t}\left[\int_0^t e^{2s}ds - \frac{3}{2}\int_0^t se^{3s}ds\right]$$

$$= c_1 e^{-3t} + e^{-3t}\left[\frac{1}{2}(e^{2t} - 1) - \frac{3}{2}\left[e^{3s}\left(\frac{s}{3} - \frac{1}{9}\right)\right]\right]$$

$$= c_1 e^{-3t} + \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} - \frac{1}{2}s + \frac{1}{6}$$

$$= c_1 e^{-3t} + \frac{1}{2}e^{-t} - \frac{1}{2}s + \frac{1}{6}$$

Similarly

$$y_2(t) = e^{-t}\int_0^t e^{s}\left(e^{-s} + \frac{3}{2}s\right)ds + c_2 e^{-t}$$

$$= \frac{3}{2}t - \frac{3}{2} + te^{-t} + \frac{3}{2}e^{-t} + c_2 e^{-t}$$

$$= c_2 e^{-t} + \frac{3}{2}(t-1) + te^{-t}$$

$$\vec{y} = \begin{pmatrix} C_1 e^{-3t} \\ C_2 e^{-t} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} e^{-t} - \frac{1}{2} s + \frac{1}{6} \\ \frac{3}{2}(t-1) + t e^{-t} \end{pmatrix}$$

$$\vec{x} = P \vec{y}.$$