Week 4

Sunday, July 14, 2019 8:53 AM

Last week we looked at $\exp(At)$.

Recall that if

$$A = PBP^{-1} \text{ then}$$

$$\exp(At) = P\exp(Bt)P^{-1}.$$

<u>Rmk</u> We only proved this where B is a diagonal matrix but it works for all matrices.

So what happens when we have a Jordan block diagonal matrix

Ie

$$A = PBP^{-1}$$

where

$$B = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ & J_2 & & \\ 0 & & \ddots & J_k \end{pmatrix}$$

where J_i are Jordan blocks

where J_ℓ are Jordan blocks.

Note

$$B^m = \begin{pmatrix} J_1^m & 0 & \cdots & 0 \\ & J_2^m & & \\ 0 & & \ddots & \\ & & & J_k^m \end{pmatrix}$$

is a block diagonal matrix.

Therefore we should have

$$\exp(Bt) = \begin{pmatrix} \exp(J_1 t) & 0 & \cdots & 0 \\ & \exp(J_2 t) & & 0 \\ 0 & & \ddots & \\ & & & \exp(J_k t) \end{pmatrix}.$$

By taking sums and limits we do get the above decomposition for $\exp(Bt)$.

So we just need to find out what $\exp(Jt)$ is for a <u>single</u> Jordan block.

Suppose $J = \Lambda + N$

where

$$\Lambda = \begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 \\ & \lambda & & & \\ 0 & & \ddots & & \end{pmatrix} = \operatorname{diag}(\lambda, \ldots, \lambda)$$

$$\Lambda = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ & \lambda & & O \\ & & \ddots & \\ O & & & \lambda \end{pmatrix} = \operatorname{diag}(\lambda, \dots, \lambda)$$

is an $n \times n$ matrix and

$$N = \begin{pmatrix} 0 & 1 & 0 & & O \\ 0 & 0 & 1 & 0 & \\ \vdots & & \ddots & \ddots & 0 \\ & & & & 1 \\ 0 & & O & & 0 \end{pmatrix}.$$

Prop] If $AB = BA$ then

$$\exp((A+B)t) = \exp(At)\exp(Bt).$$

Note $\Lambda N = N \Lambda$ since $\Lambda = \lambda I_{n \times n}$.

Thus

$$\exp((\Lambda + N)t) = \exp(\Lambda t)\exp(Nt)$$
$$= e^{\lambda t} \exp(Nt).$$

Now note

$$N^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & & \\ & & 0 & 1 & 0 & \\ & & & 0 & \ddots & 0 \\ & & & & & 1 \\ & & & & & 0 \\ & & & & & 0 \end{pmatrix}, \quad N^3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & & O \\ & & & & & \ddots & \\ & & & & & & 1 \\ & & O & & & & 0 \\ & & & & & & 0 \\ & & & & & & 0 \end{pmatrix}$$

$$N^{n-1} = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ & & & 0 \\ & & & \vdots \\ & & & 0 \end{pmatrix} \qquad N^n = 0$$

Eg $N = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$,

$$N^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad N^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$N^4 = O.$$

In general

$$\exp(Nt) = \sum_{k=0}^{\infty} \frac{1}{k!} N^k t^k$$

$$= \sum_{k=0}^{n-1} \frac{1}{k!} N^k t^k$$

$$= \begin{pmatrix} 1 & t & \frac{1}{2}t^2 & \cdots & \frac{1}{(n-1)!} t^{n-1} \\ 0 & 1 & t & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \frac{1}{2}t^2 \\ & \vdots & & & t \\ 0 & 0 & & & 1 \end{pmatrix}$$

Thus

$$\exp(Jt) = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} & \cdots & \frac{t^{n}}{(n-1)!} e^{\lambda t} \\ & \ddots & & \vdots \\ & & \ddots & \vdots \\ & O & & te^{\lambda t} \\ & & & e^{\lambda t} \end{pmatrix}$$

Existence & Uniqueness result

Thm) Suppose $A(t)$ and $\vec{b}(t)$ are a given matrix and $\vec{b}$ a vector of functions.

Consider the IVP

$$\begin{cases} \vec{x}\,' = A(t)\vec{x} + \vec{b}(t) \\ \vec{x}(t_0) = \vec{x}_0 . \end{cases}$$

If $A, \vec{b}$ are continuous (ie their entries are continuous) then there exists some small number $\varepsilon > 0$, and a vector-valued function $\vec{y}(t)$ (for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$) such that

$$\vec{y}\,' = A(t)\vec{y} + \vec{b}(t).$$

Moreover $\vec{y}$ is unique.

Methods to solve a system

Diagnolization

Consider

$$\vec{x}\,' = A\vec{x} + \vec{g}(t)$$

where A has constant entries

$\vec{x}' = A\vec{x} + \vec{g}(t)$

where A has constant entries

and is diagnolizeable.

Since A is diagnolizable by assumption

$$A = PDP^{-1}$$

for some diagnol matrix D

and change of basis P.

Then

$$\vec{x}' = PDP^{-1}\vec{x} + \vec{g}(t)$$

$$PP^{-1}\vec{x}' = PDP^{-1}\vec{x} + \vec{g}(t).$$

let $\vec{y} = P^{-1}\vec{y}$ and $\vec{h}(t) = P^{-1}\vec{g}(t)$.

then

$$P\vec{y}' = PD\vec{y} + P\vec{h}(t)$$

so $\vec{y}' = D\vec{y} + \vec{h}(t)$

$$\begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{pmatrix} = \begin{pmatrix} \lambda_1 y_1 + h_1(t) \\ \vdots \\ \lambda_n y_n + h_n(t) \end{pmatrix}$$

So you just need to solve

So you just need to solve

$$y_j' = \lambda_j y_j + h_j(t)$$

which is given by

$$y_j(t) = e^{\lambda_j t} \int_0^t e^{-\lambda_j s} h_j(s)\, ds + c_j e^{\lambda_j t}.$$

Eg

$$\vec{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

Note

$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$ has eigenvector, eigenvalues of

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}, -3 \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix}, -1$$

Let

$$P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \text{ then}$$

$$A = P \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix} P^{-1}$$

and $P^{-1} = \dfrac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

Thus

Thus

$$P\vec{y}' = PD\vec{y} + \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$$\vec{y}' = D\vec{y} + P^{-1}\begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$$= D\vec{y} + \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

Thus

$$\left.\begin{aligned} y_1' &= -3y_1 + 2 \\ y_2' &= -y_2 + 4 \end{aligned}\right\} \text{seperable and have solutions below}$$

$$y_1 = C_1 e^{-3t} + \frac{2}{3}$$

$$y_2 = c_2 e^{-t} + 4$$

Thus $\vec{y}(t) = C_1\begin{pmatrix} e^{-3t} \\ 0 \end{pmatrix} + C_2\begin{pmatrix} 0 \\ e^{-t} \end{pmatrix} + \begin{pmatrix} 2/3 \\ 4 \end{pmatrix}$

and $\vec{y} = P^{-1}\vec{x}$ so

$$\vec{x} = P\vec{y}$$

$$= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\vec{y}$$

$$= c_1 e^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 14/3 \\ 10/3 \end{pmatrix}$$

Method 2

Consider the system

$$\vec{x}'(t) = A(t)\vec{x} + \vec{b}(t).$$

And suppose you are given a fundamental set of solutions $\vec{x}_1, \dots, \vec{x}_n$ to the equation

$$\vec{x}'(t) = A(t)\vec{x}.$$

Ie $\vec{x}_j' = A(t)\vec{x}_j$ and any solution is

$$c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \cdots + c_n\vec{x}_n(t).$$

If $\Psi(t) = \begin{pmatrix} \vec{x}_1(t) & \vec{x}_2(t) & \cdots & \vec{x}_n(t) \end{pmatrix}$

then any solution to the homogeneous equation $\vec{x}'(t) = A(t)\vec{x}$ is equation $\vec{x}'(t) = A(t)\vec{x}$ is given by

$$\vec{x}(t) = \Psi(t)\vec{c}.$$

Rmk: when A is constant $\Psi(t) = e^{At}$.

Now suppose the solution to the inhomogeneous equation is given by

$$\vec{x}(t) = \Psi(t)\vec{u}(t)$$

for some function $\vec{u}(t)$.

Now note

$$\frac{d}{dt}\left[\Psi(t)\vec{u}(t)\right]$$

$$= \Psi'(t)\vec{u}(t) + \Psi(t)\vec{u}'(t)$$

$$A(t)\vec{x} + \vec{b}(t) =$$

$$A(t)\Psi(t)\vec{u}(t) + \vec{b}(t)$$

Matching terms we need

$$\Psi'(t)\vec{u}(t) = A(t)\Psi(t)\vec{u}(t)$$

and

$$\Phi(t)\,\vec{u}\,'(t) = \vec{b}(t).$$

Since Φ has column vectors which are solutions we have

$$\Phi'(t) = \begin{pmatrix} \vec{x}_1' & \vec{x}_2' & \cdots & \vec{x}_n' \end{pmatrix}$$

$$= \begin{pmatrix} A\vec{x}_1 & A\vec{x}_2 & \cdots & A\vec{x}_n \end{pmatrix}$$

$$= A(t)\,\Phi(t).$$

So the first is satisfied.

For the second we get

$$\vec{u}\,'(t) = \Phi^{-1}(t)\, b(t)$$

and integrating we get

$$\vec{u}(t) = \left[\int_c^t \Phi^{-1}(s)\,\vec{b}(s)\,ds + \underset{\substack{\uparrow \\ \text{constant vector}}}{\vec{c}} \right]$$

Thus

$$\vec{x}(t) = \Phi(t)\,\vec{u}(t)$$

$$= \Phi \left[\int^t \Phi^{-1} \cdots \right]$$

$$= \underline{\Psi}(t)\left[\int_0^t \underline{\Psi}^{-1}(s)\,\vec{b}(s)\,ds + \vec{c}\right]$$

Consider again

$$\vec{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

we know the eigenvalues & eigenvectors can give a fundamental set of solutions

$$\vec{x}_1(t) = e^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \vec{x}_2 = e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Hence

$$\underline{\Psi}(t) = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix}$$

Thus

$$\underline{\Psi}(t)\vec{u}'(t) = \vec{b}(t)$$

$$\begin{pmatrix} e^{3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} e^{3t} & e^{-t} & 6 \\ -e^{-3t} & e^{t} & 2 \end{array}\right) \qquad \text{Row reduce}$$

$$\rightsquigarrow \left(\begin{array}{cc|c} e^{3t} & e^{-t} & 6 \\ 0 & 2e^{-t} & 8 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|c} e^{3t} & e^{-t} & 6 \\ 0 & e^{-t} & 4 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|c} e^{-3t} & 0 & 2 \\ 0 & e^{-t} & 4 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 2e^{3t} \\ 0 & 1 & 4e^{t} \end{array}\right)$$

Thus

$$\left.\begin{aligned} u_1' &= 2e^{3t} \\ u_2' &= 4e^{t} \end{aligned}\right\} \Rightarrow \begin{aligned} u_1 &= \tfrac{2}{3}e^{3t} + c_1 \\ u_2 &= 4e^{t} + c_2 \end{aligned}$$

So

$$\vec{x}(t) = \Psi(t)\,\vec{u}(t)$$

$$= c_1\begin{pmatrix} e^{-3t} \\ -e^{-3t} \end{pmatrix} + c_2\begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}$$

$$+ \begin{pmatrix} 2/3 + 4 \\ -\frac{2}{3} + 4 \end{pmatrix}$$

$$\left(-\frac{2}{3} + 4 \right)$$

$$= c_1 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 14/3 \\ 10/3 \end{pmatrix}$$