Week 4 Jordan normal form example

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$$A = \frac{1}{2}\begin{pmatrix} -3 & 0 & -1 \\ 2 & -4 & -2 \\ 1 & 0 & -5 \end{pmatrix}$$

$$\det(A - \lambda I) = -(\lambda + 2)^3$$

and so the only eigenvalue is -2, and alg. mult. (-2) is 3

What is geometric mult of -2?

$$(A + 2I) = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{pmatrix}$$

which is rank 1

hence $\dim(\ker(A + 2I)) = 2$

and

$$\ker(A + 2I) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

What is a generalized eigenvector for A?

Well we need a vector, say $\vec{v}$, such that

$$(A+2I)^2\vec{v} = 0$$

$$\vec{w} = (A+2I)\vec{v} \neq 0$$

define $\vec{w}$ this way.

Looking at the first line we have

$$0 = (A+2I)^2\vec{v} = (A+2I)(A+2I)\vec{v}$$
$$= (A+2I)\vec{w}$$

and so $\vec{w} \in \ker(A+2I)$.

NB) In the Jordan basis for A, there is only one vector $\vec{v}$ which is a generalized eigenvector of rank 2. Since $\ker(A+2I)$ is 2 dimensional, but theres only 1 rank 2 generalized eigenvector there is only one dimension worth of $\vec{w} \in \ker(A+2I)$ s.t. $(A+2I)\vec{v} = \vec{w}$.

$$(A + 2I)\vec{v} = \vec{w}.$$

For example

$$(A+2I)\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{pmatrix}\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 2 & 0 & -2 & 2 \\ 1 & 0 & -1 & 0 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

which as <u>no</u> solutions

Similarly

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 2 & 0 & -2 & 0 \\ 1 & 0 & -1 & 2 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

has no solutions.

What happens here?

We tried both eigenvectors $\vec{\omega}$ and couldn't find a $\vec{v}$ s.t.

$$(A+2I)\vec{v} = \vec{\omega}$$

Let's try another way.

$$(A+2I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so lets take any vector $\vec{v}$ not in the kernel!

For example

$\vec{v}$ $\begin{pmatrix} 17 \\ -4 \\ \pi \end{pmatrix}$ is not in $\ker(A+2I)$

$$\vec{\omega} = (A+2I)\vec{v} =$$

$$\frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 17 \\ -4 \\ \pi \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 17-\pi \\ 34-2\pi \\ 17-\pi \end{pmatrix} = \frac{17-\pi}{2}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$(A+2I)\vec{w} = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{pmatrix}\left(\frac{17-\pi}{2}\right)\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$= \frac{17-\pi}{4}\begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$= \frac{17-\pi}{4}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0$$

so $\vec{w}$ is an eigvector.

Thus $\frac{17-\pi}{2}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 17 \\ -4 \\ \pi \end{pmatrix}$

is a Jordan chain.

How do we get a basis?

Well, we just need another eigenvector of A.

This is given by, for example

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Thus

$$P = \begin{pmatrix} 0 & \frac{17-\pi}{2} & 17 \\ 1 & 17-\pi & -4 \\ 0 & \frac{17-\pi}{2} & \pi \end{pmatrix}$$

is such that

$$A = PJP^{-1}$$

where

$$J = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix}.$$