Week 5, 2point boundary

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Typically when we look at differential equations and initial value problems of the form

$$\begin{cases} y'' + p(t)y' + q(t)y = g(t) \\ \left.\begin{aligned} y(t_0) = y_0 \\ y'(t_0) = y_1 \end{aligned}\right\} \text{initial values} \end{cases}$$

Physically, this means we have some physical system and we know the initial position and velocity.

This doesn't happen always we could specify the position or velocity @ 2 different times or locations. I.e.

$$\begin{cases} y'' + p(x)y' + q(x)y = g(x) \\ \left.\begin{aligned} y(\alpha) = y_0 \\ y(\beta) = y_1 \end{aligned}\right\} \text{boundary values} \end{cases}$$

$$\left.\begin{array}{l} y(\alpha) = y_0 \\ y(\beta) = y_1 \end{array}\right\} \text{boundary values}$$

Def) Consider the boundary value problem above.
If $y_0 = y_1 = 0$ and $g(t) = 0$ then we call the system homogeneous. Otherwise its nonhomogeneous.

Rmk) Under mild assumptions initial value problems have unique solutions.

However boundary value problems are not as "nice".
They can have infinitely many solutions, sometimes there are unique solutions as well.

Eg) $y'' + 2y = 0$, $y(0) = 1$, $y(\pi) = 0$

$y(x) = c_1 \cos(\sqrt{2}x) + c_2 \sin(\sqrt{2}x)$

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$$y(0) = c_1 = 1$$

$$y(\pi) = \cos(\sqrt{2}\pi) + c_2 \sin(\sqrt{2}\pi) = 0$$

$$c_2 = \frac{\cos(\sqrt{2}\,\pi)}{\sin(\sqrt{2}\,\pi)} = \cot(\sqrt{2}\pi)$$

Hence there is a unique solution given by

$$y(x) = \cos(\sqrt{2}\,x) + \cot(\sqrt{2}\pi)\sin(\sqrt{2}\,x)$$

Eg) $y'' + 4y = 0, \quad y(0) = y(\pi) = 0$

$y(x) = a\sin(2x)$ solves this for any $a \in \mathbb{R}$.

Hence there are ∞ many solutions

Eg) $y'' + 2y = 0 \quad y(0) = y(\pi) = 0$

$$y(x) = c_1 \cos(\sqrt{2}\,x) + c_2 \sin(\sqrt{2}\,x)$$

$y(x) = c_1 \cos(\sqrt{2}\,x) + c_2 \sin(\sqrt{2}\,x)$

$y(0) = c_1 = 0$

$y(\pi) = c_2 \underbrace{\sin(\sqrt{2}\,\pi)}_{\neq 0} = 0$ and so

the <u>only</u> solution is $y \equiv 0$.

Now consider in general

$$\begin{cases} y'' + \lambda y = 0 \\ y(0) = 0 \\ y(\pi) = 0 \end{cases} \qquad (\ast)$$

By extending the terminology from linear algebra

we'll say λ is an <u>eigenvalue</u> if there is some non-zero solution to $(\ast)$.

<u>Eg</u>) 4 is an eigenvalue

2 is <u>not</u>

<u>HW</u>) Show that the only eigenvalues are $\lambda = n^2$ for some integer $n \neq 0$

eigenvalues are $\lambda = n$ for some integer $n \neq 0$.

We call the non-trivial solutions to ⊗ eigenfunctions w/ eigenvalue λ.

In general

$$\begin{cases} y'' + \lambda y = 0 \\ y(0) = 0 \\ y(L) = 0 \end{cases}$$

has eigenvalues and eigenfunctions
$n = 1, 2, \ldots$

$$\lambda_n = \frac{n^2 \pi^2}{L^2}, \quad y_n(x) = \sin\left(\frac{n\pi}{L} x\right)$$

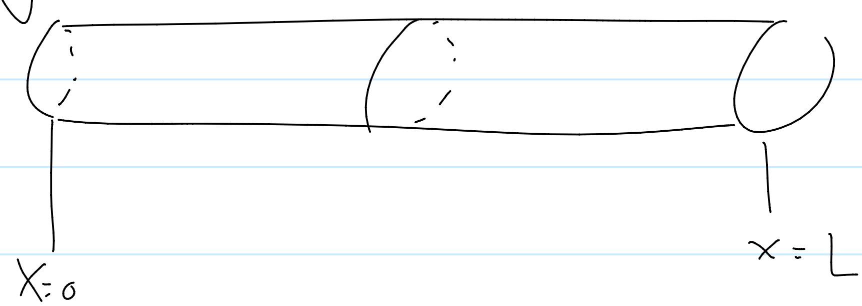
An idea above is that there can be

- no solutions
- 1 solution

· ∞ many solutions when we have boundary value data instead of initial values.

Let's move on to one of the main PDEs we'll study in the second half.

Suppose you have a bar of length L.



Let $u = u(x,t)$ denote the temperature of the bar at time $t \geq 0$ and position x.

It turns out that there's a "simple" PDE that describes u.

$$u_t = \alpha^2 u_{xx} \qquad 0<x<L, \quad t>0$$

$$u_t = \alpha^2 u_{xx} \qquad 0 < x < L \quad , t > 0$$

where α depends on physical properties of the bar.

The bar has some temperature profile at time $t = 0$ and this means

$$u(x,0) = f(x) \qquad 0 \leq x \leq L$$

$\uparrow$ initial temperature

We also assume that the temperature at the ~~ends~~ ends of the bar are fixed (and we assume the fixed temperature is 0).

$$u(0,t) = u(L,t) = 0 \quad . \; t > 0.$$

<u>Method of separation of variables</u>

We want to find solutions to

$$u_t = \alpha^2 u_{xx}$$

to

$$\begin{cases} u_t = \alpha^2 u_{xx} \\ u(x,0) = f(x) \\ u(0,t) = u(L,t) = 0 \end{cases}$$

We start by <u>assuming</u> that

$$u(x,t) = X(x)\,T(t)$$

is the product of two functions

We'll omit x, t from what follows

$$u_t = X T'$$
$$u_{xx} = X'' T$$

and so

$$X T' = \alpha^2 X'' T$$

impliying

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T}$$

only a function

only a function of x — only a function of t.

Note if $g(x) = h(t)$

$0 < x < L$, $t > 0$

then $\frac{\partial}{\partial t} g(x) = 0 = \frac{\partial}{\partial t} h(t) = h'(t)$

$$\frac{\partial}{\partial x} g(x) = g'(x) = \frac{\partial}{\partial x} h(t) = 0.$$

So $g'(x) = 0$, $h'(t) = 0$ and so they are <u>constant</u>.

So there exists some number such that

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = -\lambda$$

So

$$X'' + \lambda X = 0$$
$$T' + \alpha^2 \lambda T = 0$$

What are the λ [illegible] values here?

What are the boundary values here?

Well $u(0,t) = 0$ and

$$\parallel$$

$$X(0)T(t) = 0$$

So either $T(t) = 0$ for all t or

$X(0) = 0$. [$T(t) = 0$ means $u(x,t) = 0$ which is boring so we assume $X(0) = 0$]

Similarly $X(L) = 0$.

Thus

$$X'' + \lambda X = 0, \quad X(0) = X(L) = 0$$

meaning the only nonzero solutions are

$$X_n(x) = \sin\left(\frac{\pi n}{L} x\right) \quad \text{w/ eigenvalue}$$

$$\lambda_n = \frac{\pi^2 n^2}{L^2}$$

Turning to the solution for T we then have

$$T' + \frac{\pi^2 n^2 \alpha^2}{L^2} T = 0$$

meaning

$$T(t) = \exp\left(\frac{-n^2 \pi^2 \alpha^2 t}{L^2}\right)$$

and so

$$u_n(x,t) = e^{-\lambda_n \alpha^2 t} \sin\left(\frac{n\pi}{L} x\right)$$

solves

$$u_t = \alpha^2 u_{xx} \quad , \quad u(0,t) = u(L,t) = 0$$

but not $u(x,0) = f(x)$.

<u>Prop</u> (Principle of superposition)

If u and v solve

$$u_t = \alpha^2 u_{xx}$$

then $u+v$ also does. ☒

Hence

$$\sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n \alpha^2 t} \sin\left(\frac{\pi n}{L} x\right)$$

solves the PDE

and if

$$\sum_{n=1}^{\infty} c_n \sin\left(\frac{\pi n}{L} x\right) = f(x)$$

for some c_n then we can solve the original PDE.