End of Week 6, Start of Week 7 - Fourier

Friday, August 2, 2019 6:39 AM

Wednesday, and before, we have shown that certain PDE's can be solved by expanding a function $f(x)$, the initial temperature, into an infinite sum of sin series.

Eg) We have to write

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{2} x\right)$$

where

$$c_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{2} x\right) dx.$$

The reason we looked at sine curves for

$$\begin{cases} u_t = \alpha^2 u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

$$u(0,t) = u(L,t) = 0$$
$$u(x,0) = f(x)$$

is because the eigenvalue problem

$$\begin{cases} y'' + \lambda y = 0 \\ y(0) = y(L) = 0 \end{cases}$$

has eigenfunctions of

$\sin\left(\frac{n\pi}{L}x\right)$.

Recall from HW 5

$$\begin{cases} y'' + \lambda y = 0 \\ y'(0) = y'(\pi) = 0 \end{cases}$$

has

eigenfunctions

$$y(x) = 1$$
$$y(x) = \cos\left(\frac{n\pi}{L}x\right).$$

So it may be useful to expand a function into both sines and cosines.

In order to do this

we have to establish some notation and some assumptions.

Note

$$\cos\left(\frac{n\pi}{L}(x+2L)\right) = \cos\left(\frac{n\pi}{L}x + 2n\pi\right)$$
$$= \cos\left(\frac{n\pi}{L}x\right)$$

$$\sin\left(\frac{n\pi}{L}(x+2L)\right) = \sin\left(\frac{n\pi}{L}x + 2n\pi\right)$$
$$= \sin\left(\frac{n\pi}{L}x\right)$$

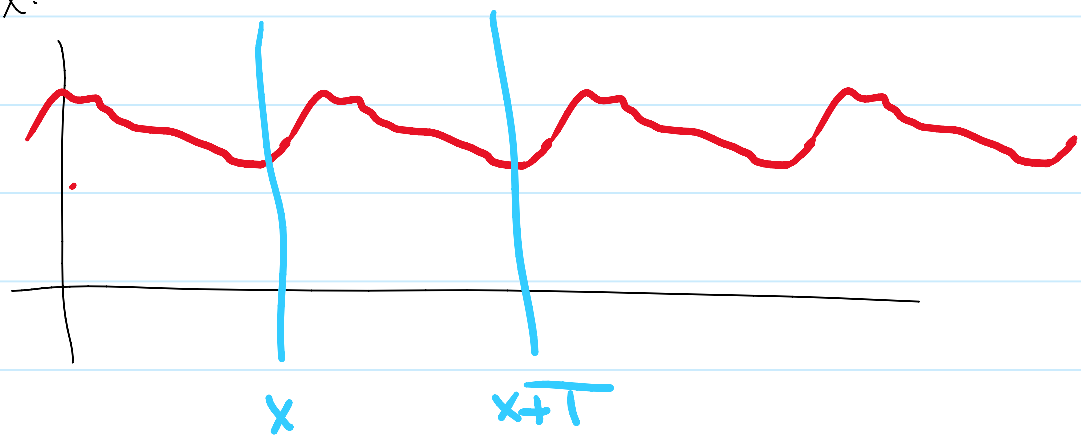
and so if

$$(*)\quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)$$

then $f(x+2L) = f(x)$.

So if $f(x)$ can be expanded into the infinite sum in $(*)$ we need f to be periodic.

Def A function f is <u>periodic</u> with <u>period T</u> (also called <u>T-periodic</u>) if $f(x+T) = f(x)$ for every x.



Eg The following functions are $2L$ periodic

$\cos\left(\frac{2\pi}{L}x + 7\right)$, $\sin\left(\frac{\pi}{L}x - \pi\right)$

$17 + \sin\left(\frac{2\pi}{L}x\right)$

Notation

We let $\mathcal{P}_T$ denote all T-periodic functions.

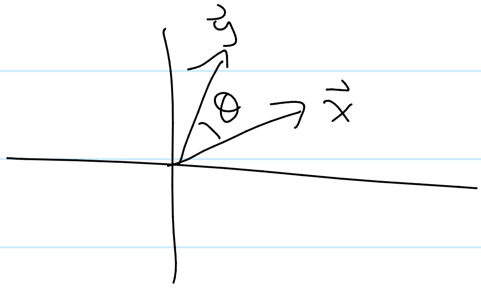
Note if a, b are constants and f, g are T-periodic functions then

$$(af+bg)(x+T)$$
$$= af(x+T) + bg(x+T)$$
$$= af(x) + bg(x)$$
$$= (af+bg)(x).$$

And so $\mathcal{P}_T$ is a <u>vector space</u>.
(since $y(x)=0$ is T-periodic too).

Whats another vector space we know of?
$\mathbb{R}^n$.

And $\mathbb{R}^n$ has an additional property. We can define "angles" between non-zero vectors $\vec{x}, \vec{y}$.



How do we compute θ?

How do we computed Θ?

Recall the dot-product which we'll write as $(\vec{x}, \vec{y})$.

For vectors in $\mathbb{R}^n$

$$(\vec{x}, \vec{y}) = \sum_{1}^{n} x_j y_j .$$

and

$$\cos \theta = \frac{(\vec{x}, \vec{y})}{|\vec{x}| \; |\vec{y}|}, \quad |\vec{x}| = \sqrt{(\vec{x}, \vec{x})}.$$

Def) We say two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$ are <u>orthogonal</u> if $(\vec{x}, \vec{y}) = 0$.

Note if

$$e_j = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad j^{th} \text{ position}$$

the $(\vec{x}, \vec{e}_j) = x_j$ if

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

And so

$$\vec{x} = \sum^{n} (\vec{x}, e_i) \, \vec{e}_i .$$

$$\vec{x} = \sum_{j=1}^{n} (\vec{x}, e_j)\, \vec{e}_j .$$

Why did I just spend time discussing $\mathbb{R}^n$? Well, the space $\mathcal{P}_{2L}$ also has a nice "dot product", which we call an <u>inner-product</u>. in this case.

<u>Def</u> On the vector space $\mathcal{P}_{2L}$, the $2L$-periodic functions, we define the <u>inner product</u> of f, g as

$$(f, g) = \frac{1}{L} \int_{-L}^{L} f(x)\, g(x)\, dx .$$

and we define the <u>norm</u> of a function f as

$$\|f\|_2 = \sqrt{(f,f)}$$

$$= \left(\frac{1}{L}\int_{-L}^{L} f(x)^2\, dx\right)^{1/2}.$$

<u>Note!</u> On HW 5 you showed the following facts

$$\frac{1}{L}\int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx = 0$$

$$\frac{1}{L}\int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx$$

$$= \frac{1}{L}\int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx$$

$$= \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

and

$$\frac{1}{L}\int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right) = \frac{1}{L}\int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right)dx = 0.$$

This means the functions

$$\cos\left(\frac{n\pi}{L}x\right), \sin\left(\frac{n\pi}{L}x\right), \underline{1}$$

are <u>orthogonal</u>.