Week 7: More Heat Conductance

Wednesday, August 7, 2019 8:17 AM

So far, we have only solved

$$\begin{cases} u_t = \alpha^2 u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

and we found a solution

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\alpha^2 n^2 \pi^2 t/L^2} \sin\left(\frac{n\pi}{L} x\right)$$

$$c_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx.$$

What happens if the ends of the rod are <u>not</u> at zero temperature?

$$\begin{cases} u_t = \alpha^2 u_{xx} \\ u(0,t) = T_1 \\ u(L,t) = T_2 \\ u(x,0) = f(x). \end{cases}$$

Lets write

$$v(x,t) = T_1 + \frac{(T_2 - T_1)x}{L}$$

Now $v(0,t) = T_1$, $v(L,t) = T_2$.

What else?

$$V_{xx} = 0 \quad \text{and} \quad v_t = 0$$

So v solves:

$$\begin{cases} v_t = \alpha^2 v_{xx} \\ v(0,t) = T_1 \\ v(L,t) = T_2 \\ v(x,0) = T_1 + (T_2 - T_1)\frac{x}{L}. \end{cases}$$

So what does

$$w(x,t) = u(x,t) - v(x,t)$$

Note

$$\alpha^2 w_{xx} = \alpha^2 u_{xx} - \alpha^2 v_{xx}$$

$$= u_t - v_t$$

$$= w_t \,.$$

$w(0,t) = 0$, $w(L,t) = 0$

and

$$w(x,0) = \underbrace{f(x) - T_1 - (T_2 - T_1)\frac{x}{L}}_{\text{call this } g(x).}.$$

Then w solves:

$$\begin{cases} w_t = \alpha^2 w_{xx} \\ w(0,t) = w(L,t) = 0 \\ w(x,0) = g(x) \end{cases}$$

So

$$w(x,t) = \sum_{n=1}^{\infty} c_n e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t} \sin\left(\frac{n\pi}{L} x\right)$$

$$c_n = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

Moreover

$$w(x,t) = u(x,t) - v(x,t)$$

So

$$u(x,t) = T_1 + (T_2 - T_1)\frac{x}{L}$$

$$u(x,t) = T_1 + (T_2 - T_1)\frac{x}{L} + \sum_{n=1}^{\infty} C_n e^{-\alpha^2 n^2 \pi^2 t/L^2} \sin\left(\frac{n\pi}{L}x\right)$$

$$C_n = \frac{2}{L}\int_0^L \left[f(x) - T_1 - \frac{(T_2 - T_1)x}{L}\right] \sin\left(\frac{n\pi}{L}x\right) dx$$

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omitting the t.

$$u(x,t) = w(x,t) + v(x)$$

Now note

$$\lim_{t\to\infty} e^{-\alpha^2 n^2 \pi^2 t/L^2} = 0$$

so

$$\lim_{t\to\infty} w(x,t) = \lim_{t\to\infty} \sum_{n=1}^{\infty} C_n e^{-\alpha^2 n^2 \pi^2 t/L^2} \sin\left(\frac{n\pi}{L}x\right)$$

$$= 0$$

But $\lim_{t\to\infty} v(x) = v(x)$

So

$$\lim_{t \to \infty} u(x,t) = v(x) = T_1 + (T_2 - T_1)\frac{x}{L}$$

Eg)
$$\begin{cases} u_t = u_{xx} \\ u(0,t) = 20, \quad u(30,t) = 50 \\ u(x,0) = 60 - 2x \end{cases}$$

$$v(x) = 20 + \frac{(50-20)x}{30} = 20 + x.$$

$$\begin{aligned} g(x) &= f(x) - (20+x) \\ &= (60-2x) - (20+x) \\ &= 40 - 3x \end{aligned}$$

The function $w(x,t)$ solves
$$\begin{cases} w_t = w_{xx} \\ w(0,t) = w(30,t) = 0 \\ w(x,0) = 40 - 3x \end{cases}$$

$$w(x,t) = \sum_{n=1}^{\infty} c_n e^{-\frac{n^2\pi^2}{900}t} \sin\left(\frac{n\pi}{30}x\right)$$

where

$$C_n = \frac{2}{L}\int_0^L (40-3x)\sin\left(\frac{n\pi}{30}x\right)dx$$

$$= \frac{300}{\pi n}\left(5(-1)^n + 4\right)\cdot\frac{2}{30}$$

$$= \frac{20}{\pi}\left(\frac{5(-1)^n + 4}{n}\right)$$

What about when both ends of the rod are insulated? Ie heat cannot pass through the ends of the bars. this means we have the following PDE

$$\begin{cases} u_t = \alpha^2 u_{xx} \\ u(x,0) = f(x) \\ u_x(0,t) = u_x(L,t) = 0 \end{cases}$$

change in temperature on the ends of the bars.

Again to solve this we use separation of variables:

Assume

$$u(x,t) = X(x)\,T(t) = XT.$$

and then

$$\left.\begin{aligned} u_{xx} &= X''T \\ u_t &= XT' \end{aligned}\right\} \Rightarrow XT' = \alpha^2 X''T$$

$$\text{So} \quad \frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = -\lambda$$

function of x = function of t $\Rightarrow$ they are constant

$-\lambda$ is called the separation constant on the homework.

We solve:

$$X'' = -\lambda X$$

$$X'(0) = u_x(0,t)/T = 0$$

$$X'(L) = \frac{u_x(L,t)}{T} = 0$$

So we solve

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = X'(L) = 0 \end{cases}$$

by HW 5 we know the eigenvalue and eigenfunctions are

eigenvalue	eigenfunction
0	1
$\frac{n^2\pi^2}{L^2}$	$\cos\left(\frac{n\pi}{2} x\right)$

For these λ_n the solutions to $T' = -\alpha^2 \lambda_n T$ are

λ_n	T
0	constant
$\frac{n^2\pi^2}{L^2}$	$\text{constant} \cdot \exp\left(\frac{-\alpha^2 n^2 \pi^2}{L^2} t\right)$

Thus the solution to

$$\begin{cases} u_t = \alpha^2 u_{xx} \\ u_x(0,t) = u_x(L,t) \\ \quad = 0 \end{cases}$$

$$\begin{cases} u_x(0,t) = u_x(L,t) \\ u(x,0) = f(x) \end{cases}$$

is

$$u(x,t) = c_0 + \sum_{n=1}^{\infty} c_n e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t} \cos\left(\frac{n\pi}{L} x\right)$$

and so

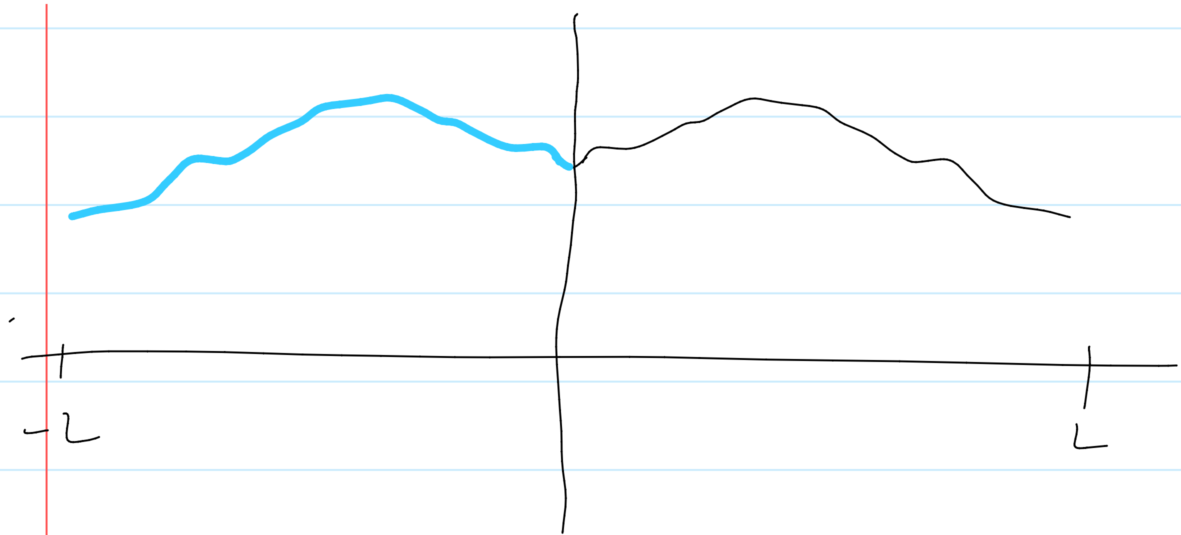
$$u(x,0) = c_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi}{L} x\right)$$

where

$$c_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

$$c_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi}{L} x\right) dx$$

These coefficients are the Fourier coefficients for the even extension of $f(x)$.



The black curve on the right is the original $f(x)$, the blue & black curves together are the even extension of f.