Week 7 Pt 1

Monday, August 5, 2019 6:52 AM

On Friday we discussed periodic functions, We defined $\mathcal{P}_T$ as the collection of T-periodic functions ($f(x+T) = f(x)$ for all x).

$\mathcal{P}_T$ is a vector space since $af+g$ is T-periodic if f and g are T-periodic.

We also briefly discussed the inner product on $\mathcal{P}_{2L}$.

Recall

$$(f,g) = \frac{1}{L}\int_{-L}^{L} f(x)g(x)\,dx.$$

Under this we have the following facts

$$\left(\cos\left(\frac{n\pi}{L}x\right), \sin\left(\frac{m\pi}{L}x\right)\right) = 0 \qquad m,n \geq 1$$

$$\left(1, \cos\left(\frac{n\pi}{L}x\right)\right) = 0 \qquad n \geq 1$$

$$\left(1, \sin\left(\frac{n\pi}{L}x\right)\right) = 0 \qquad n \geq 1$$

$$\left(\cos\left(\frac{n\pi}{L}x\right), \cos\left(\frac{m\pi}{L}x\right)\right) = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

$$\left(\sin\left(\frac{n\pi}{L}x\right), \sin\left(\frac{m\pi}{L}x\right)\right) = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

$$(1, 1) = 2.$$

The inner product on $\mathcal{P}_{2L}$ is <u>similar</u> to the dot product on $\mathbb{R}^n$.

Let's look at this $\mathbb{R}^4$ example

Here is a basis of $\mathbb{R}^4$:

$$\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\vec{v}_1, \quad \vec{v}_2, \quad \vec{v}_3, \quad \vec{v}_4.$$

Note $\vec{v}_i \cdot \vec{v}_j = 0$ for each $i \neq j$ so these vectors are orthogonal.

So these vectors are orthogonal.

How do we write $\vec{x} = \begin{pmatrix} 2 \\ 7 \\ 0 \\ 3 \end{pmatrix}$ as

a linear combination of these vectors?

Using the fact $\vec{v}_1, \dots, \vec{v}_4$ form

a basis we can say

$$\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 + c_4 \vec{v}_4 .$$

What is c_1? We'll write $(\vec{x}, \vec{v}) = \vec{x} \cdot \vec{v}$.

Let's look at

$$(\vec{x}, \vec{v}_1) = (c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 + c_4 \vec{v}_4, \vec{v}_1)$$

$$= c_1 (\vec{v}_1, \vec{v}_1) + c_2 (\vec{v}_2, \vec{v}_1) + c_3 (\vec{v}_3, \vec{v}_1) + c_4 (\vec{v}_4, \vec{v}_1)$$

$$= c_1 (\vec{v}_1, \vec{v}_1) + 0 + 0 + 0$$

So $$c_1 = \frac{(\vec{x}, \vec{v}_1)}{(\vec{v}_1, \vec{v}_1)} = \frac{-8}{4} = -2$$

We can do the same with the other c_j's.

$$c_2 = \frac{(\vec{x}, \vec{v}_2)}{(\vec{v}_2, \vec{v}_2)} = \frac{12}{4} = 3$$

$$c_3 = \frac{(\vec{x}, \vec{v}_3)}{} = \frac{-2}{} = -1$$

$$c_3 = \frac{(\vec{x}, \vec{v}_3)}{(\vec{v}_3, \vec{v}_3)} = \frac{-2}{2} = -1$$

$$c_4 = \frac{(\vec{x}, \vec{v}_4)}{(\vec{v}_4, \vec{v}_4)} = \frac{4}{2} = 2$$

So

$$\begin{pmatrix} 2 \\ 7 \\ 0 \\ 3 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - 1 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 + 3 - (-1) \\ -(-2) + 3 + 2 \\ -2 + 3 - 1 \\ -(-2) + 3 - 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \\ 0 \\ 3 \end{pmatrix}.$$

This works in general:

<u>Prop</u> Suppose $\vec{v}_1, \dots, \vec{v}_n$ are <u>any</u> n orthogonal vectors in $\mathbb{R}^n$.

(ie $v_j \cdot v_i = 0$ for $j \neq i$)

Write $(\vec{v}, \vec{w})$ for $\vec{v} \cdot \vec{w}$.

Let $\vec{x}$ be <u>any</u> vector in $\mathbb{R}^n$.

Then

$$\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n$$

where

$$c_j = \frac{(\vec{x}, \vec{v}_j)}{(\vec{v}_j, \vec{v}_j)}.$$

How does this generalize?

Well recall the inner products for sin and cos above?

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)$$

any 2L periodic function

where $$a_0 = \frac{(f, \frac{1}{2})}{(\frac{1}{2}, \frac{1}{2})} = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{\left(f, \cos\left(\frac{n\pi}{L}x\right)\right)}{\left(\cos\left(\frac{n\pi}{L}x\right), \cos\left(\frac{n\pi}{L}x\right)\right)} = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx$$

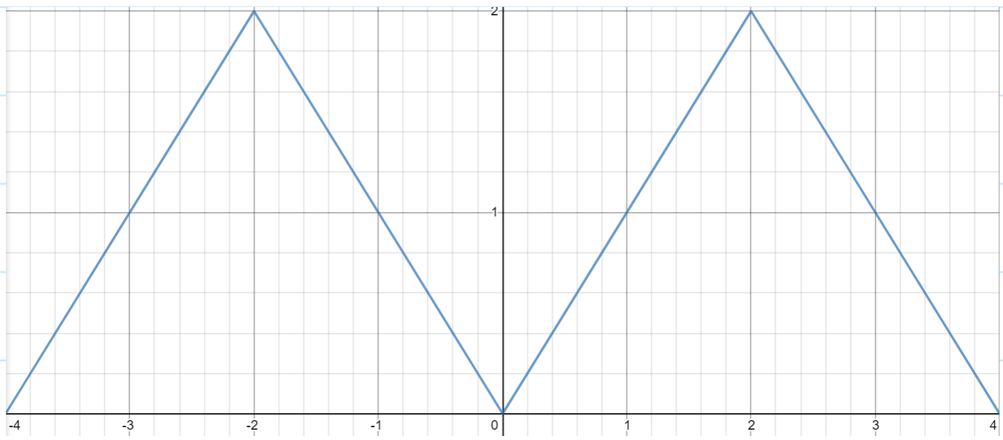
$$b_n = \frac{\left(f, \sin\left(\frac{n\pi}{L}x\right)\right)}{\left(\sin\left(\frac{n\pi}{L}x\right), \sin\left(\frac{n\pi}{L}x\right)\right)} = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

Ex

Consider $f(x) = |x|$ for $-2 \le x \le 2$

and $f(x+4) = f(x)$.

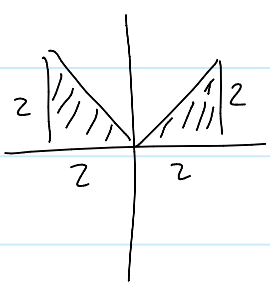
Graph of f. and it repeats.



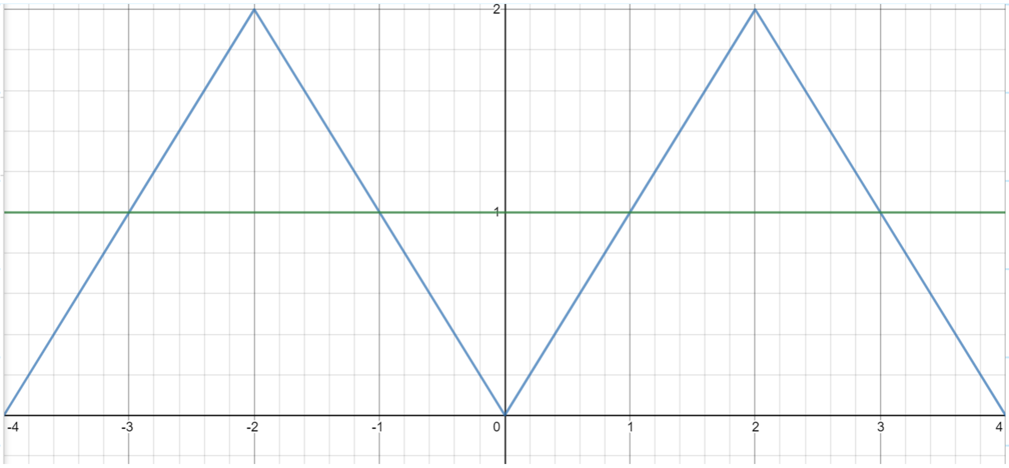
Now let's compute some coefficients.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{2} x\right) + b_n \sin\left(\frac{n\pi}{2} x\right).$$

$$a_0 = \frac{1}{2} \int_{-2}^{2} f(x)\,dx = \int_{-2}^{2} |x|\,dx = 2$$



So the first "approximation" to f is $f(x) = \frac{a_0}{2} = 1$.



It's not that good.

Integration by parts can show

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi}{2}x\right)dx$$

$$= \begin{cases} -8/(n\pi)^2 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$b_n = \frac{1}{2}\int_{-2}^{2} |x| \sin\left(\frac{n\pi}{2}x\right) dx$$

$$= \frac{1}{2}\int_{-2}^{0} -x \sin\left(\frac{n\pi}{2}x\right)dx + \frac{1}{2}\int_{0}^{2} x\sin\left(\frac{n\pi}{2}x\right)dx$$

$$= \frac{1}{2}\int_{2}^{0} y \sin\left(\frac{n\pi}{2}\cdot -y\right)\cdot -1\, dy + \frac{1}{2}\int_{0}^{2} x \sin\left(\frac{n\pi}{2}x\right)dx$$

$$= \frac{1}{2}\int_{0}^{2} -y \sin\left(\frac{n\pi}{2}y\right)dy + \frac{1}{2}\int_{0}^{2} y \sin\left(\frac{n\pi}{2}y\right)dy$$

$= 0$.

Thus

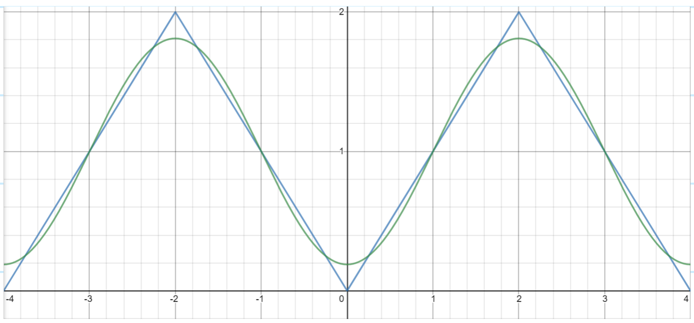
$$f(x) = 1 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{2} x\right)$$

$$= 1 + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos\left(\frac{(2n-1)\pi x}{2}\right)$$

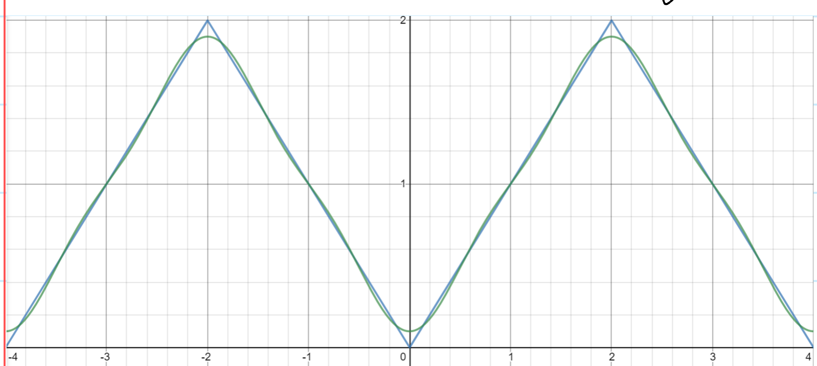
Let's look at some partial sums:

$$1 - \frac{8}{\pi^2} \cos\left(\frac{(2-1)\pi}{2} x\right)$$

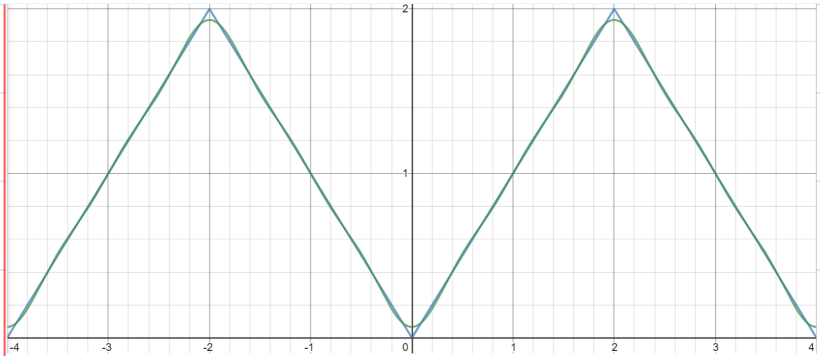
$$= 1 - \frac{8}{\pi^2} \cos\left(\frac{\pi x}{2}\right)$$



$$1 - \frac{8}{\pi^2} \cos\left(\frac{1}{2}\pi x\right) - \frac{8}{\pi^2} \frac{1}{9} \cos\left(\frac{3\pi}{2} x\right)$$

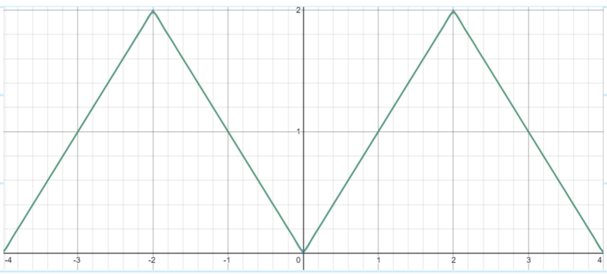


$$1 - \frac{8}{\pi^2} \cos\left(\frac{1}{2}\pi x\right) - \frac{8}{9\pi^2} \cos\left(\frac{3}{2}\pi x\right) - \frac{8}{25\pi^2} \cos\left(\frac{5}{2}\pi x\right)$$

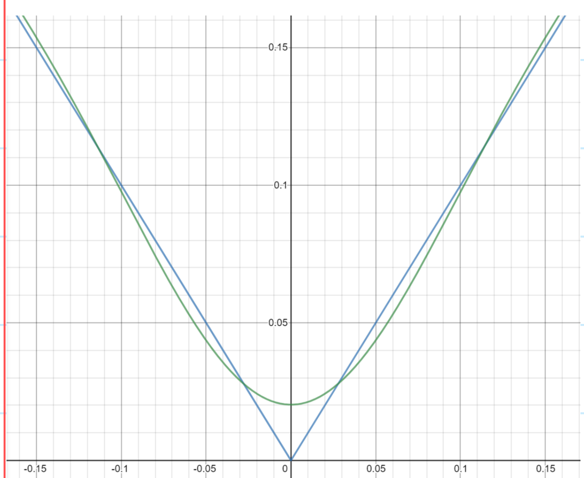


Going to better approximations:

$$1 - \frac{8}{\pi^2} \sum_{n=1}^{10} \frac{1}{(2n-1)^2} \cos\left(\frac{(2n-1)\pi}{2} x\right)$$



They are "practically" the same curve.



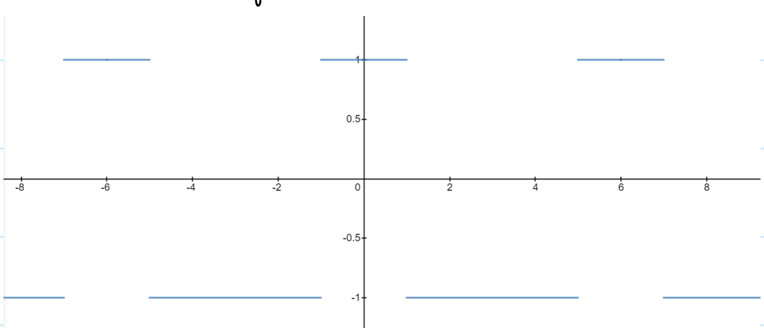
They differ by about 0.02.

What about discontinuous functions that are periodic?

$$f(x) = \begin{cases} -1 & -3 < x < -1 \\ 1 & -1 < x < 1 \\ -1 & 1 < x < 3 \end{cases}$$

and $f(x+6) = f(x)$

Here's a graph of f:



Let's compute some terms

$$a_0 = \frac{1}{3}\int_{-3}^{3} f(x)\,dx = \frac{1}{3}(-2+2-2)$$
$$= -\frac{2}{3}$$

What about a_n and b_n?

Well $f(x) = f(-x)$ so,

as with the example above

as with the example abo[v]e

$$\int_{-3}^{3} f(x) \sin\left(\frac{n\pi}{3}x\right) dx = 0.$$

What about a_n?

$$3a_n = \int_{-3}^{3} f(x) \cos\left(\frac{n\pi}{3}x\right) dx$$

$$= \int_{-3}^{-1} -\cos\left(\frac{n\pi}{3}x\right) dx + \int_{-1}^{1} \cos\left(\frac{n\pi}{3}x\right) - \int_{1}^{3} \cos\left(\frac{n\pi}{3}x\right) dx$$

$$\int_a^b \cos\left(\frac{n\pi}{3}x\right) dx = \frac{3}{n\pi} \sin\left(\frac{n\pi a}{3}\right) - \frac{3}{n\pi} \sin\left(\frac{n\pi}{3}b\right).$$

So

$$-\int_{-3}^{-1} \cos\left(\frac{n\pi}{3}x\right) dx = \frac{-3}{n\pi}\left(\sin\left(-\frac{n\pi}{3}\right) - \sin(n\pi)\right)$$

$$= \frac{3}{n\pi} \sin(n\pi/3)$$

$$\int_{-1}^{1} \cos\left(\frac{n\pi}{3}x\right) dx = \frac{3}{n\pi}\left(\sin\left(\frac{n\pi}{3}\right) - \sin\left(\frac{-n\pi}{3}\right)\right)$$

$$= \frac{6}{n\pi} \sin(n\pi/3)$$

$$-\int_{1}^{3} \cos\left(\frac{n\pi}{3}x\right) dx = \frac{-3}{n\pi}\left(\sin(n\pi) - \sin\left(\frac{n\pi}{3}\right)\right)$$

$$= \frac{3}{} \sin\left(\frac{n\pi}{}\right)$$

$$= \frac{3}{n\pi} \sin\left(\frac{n\pi}{3}\right)$$

Thus

$$a_n = \frac{1}{3}\left(\frac{3}{n\pi} + \frac{6}{n\pi} + \frac{3}{n\pi}\right) \sin\left(\frac{n\pi}{3}\right)$$

$$= \frac{4}{n\pi} \sin\left(\frac{n\pi}{3}\right)$$

Therefore

$$f(x) = \frac{-1}{3} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{3}\right) \cos\left(\frac{n\pi}{3}x\right)$$

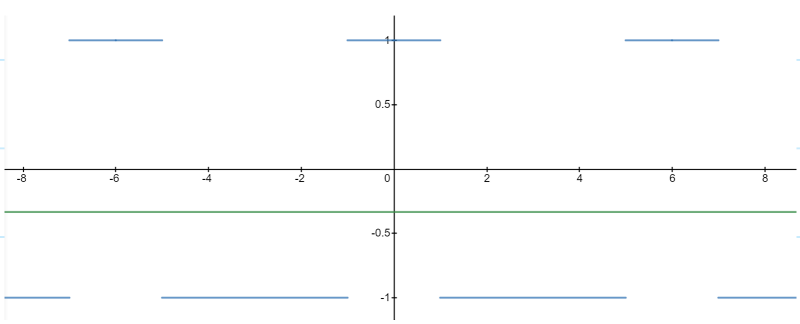
$$= \frac{-1}{3} + \frac{2\sqrt{3}}{\pi} \cos\left(\frac{\pi}{3}x\right) + \frac{\sqrt{3}}{\pi} \cos\left(\frac{2\pi}{3}x\right)$$

$$- \frac{\sqrt{3}}{2\pi} \cos\left(\frac{4\pi}{3}x\right) - \frac{2\sqrt{3}}{5\pi} \cos\left(\frac{5\pi}{3}x\right) + \cdots$$

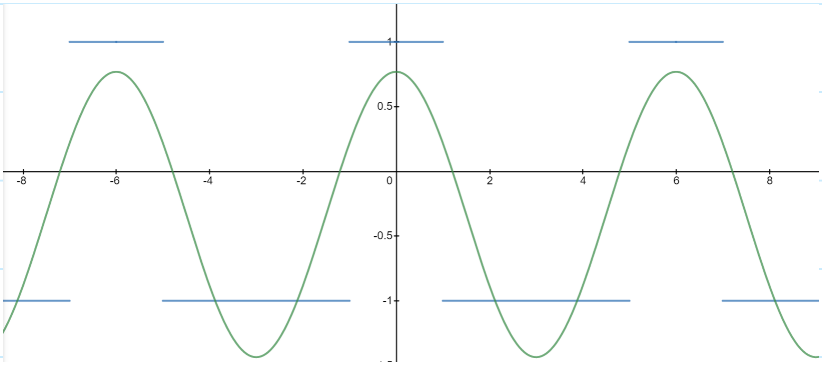
Here are the first few partial sums of

$$f(x) = \frac{-1}{3} + \frac{4}{\pi} \sum_{n=1}^{N} \frac{1}{n} \sin\left(\frac{n\pi}{3}\right) \cos\left(\frac{n\pi}{3}x\right)$$

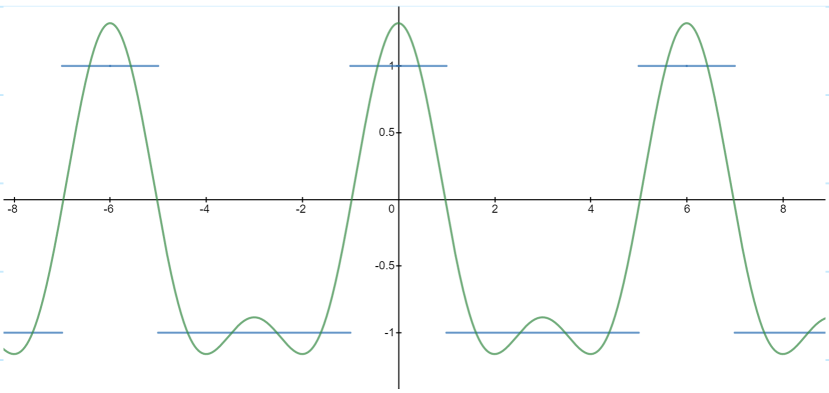
$N = 0$



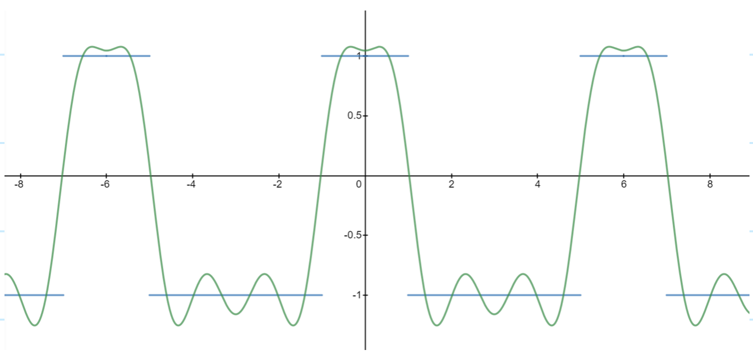
N=1



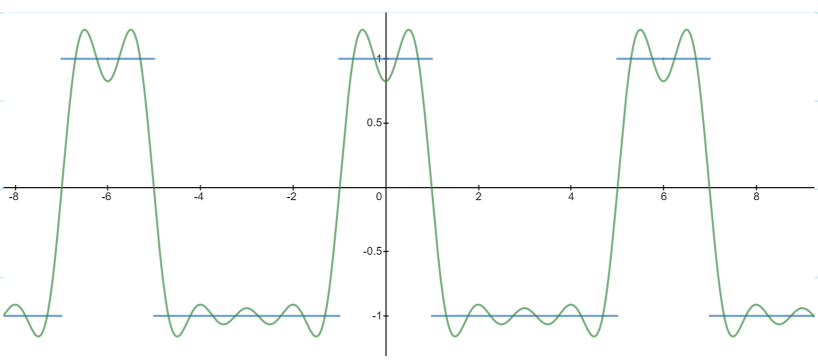
N=2 and N=3



N=4



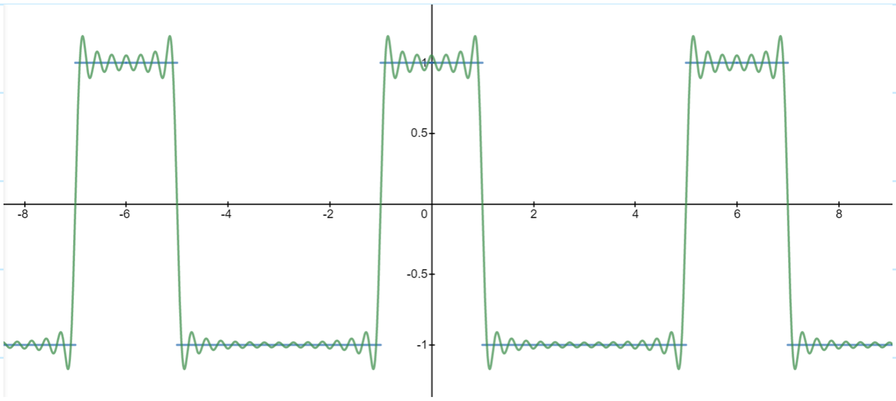
N=5



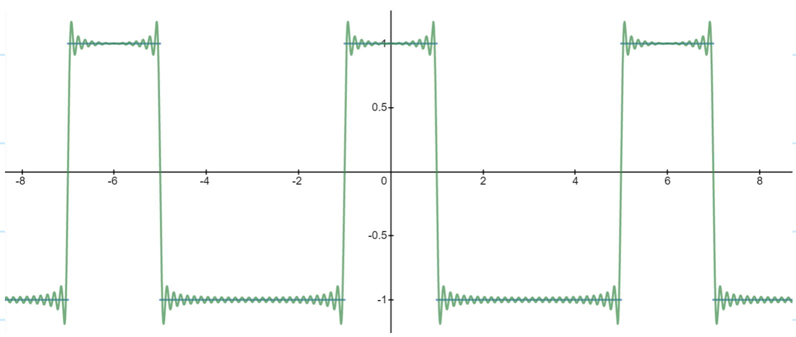
The partial sums are getting better

The partial sums are getting better
let's jump ahead to a larger N.

$N=20$ ↳



$N=40$ ↳



Question:
Will the partial sums converge to $f(x)$? I.e. as $N \to \infty$ will

$$f(x) = \lim_{N\to\infty} -\frac{1}{3} + \frac{4}{\pi}\sum_{n=1}^{N} \frac{\sin\left(\frac{n\pi}{3}\right)}{n}\cos\left(\frac{n\pi}{3}x\right) ?$$

Answer:

as long as

$x \neq \ldots, -7, -5, -3, -1, 1, 3, 5, \ldots$

the limit will converge to $f(x)$.

Why?

$\boxed{\text{Thm}}$ (Fourier convergence theorem)

Suppose f and f' are piecewise continuous functions on $-L \leq x \leq L$, and is $2L$-periodic.

Then with a_0, a_n, b_n defined above let

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)$$

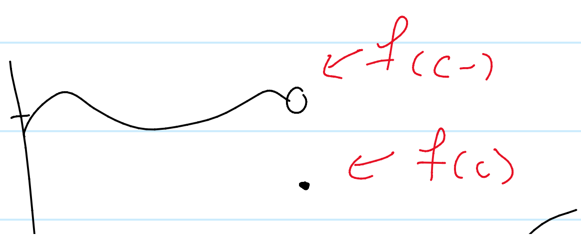
Then

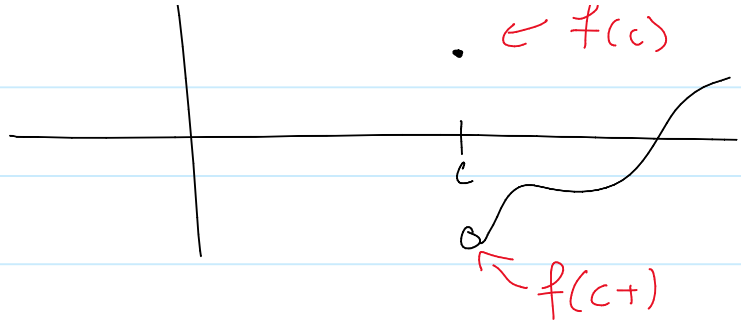
$$\lim_{N \to \infty} S_N(x_0) = \begin{cases} f(x_0) & f \text{ is continuous at } x_0 \\ \dfrac{f(x_0+) + f(x_0-)}{2} & f \text{ is discontinuous at } x_0 \end{cases}$$

Here $f(c+) = \lim_{x \to c^+} f(x)$

$f(c-) = \lim_{x \to c^-} f(x)$.

I.e.





Take away:

Every 2L-periodic function f has a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right).$$