Week 7 pt 2

Wednesday, August 7, 2019 6:56 AM

$\boxed{\text{Thm}}$ (Fourier convergence theorem)

Suppose f and f' are piecewise continuous functions on $-L \leq x \leq L$, and is $2L$-periodic.

Then with a_0, a_n, b_n defined above let

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)$$

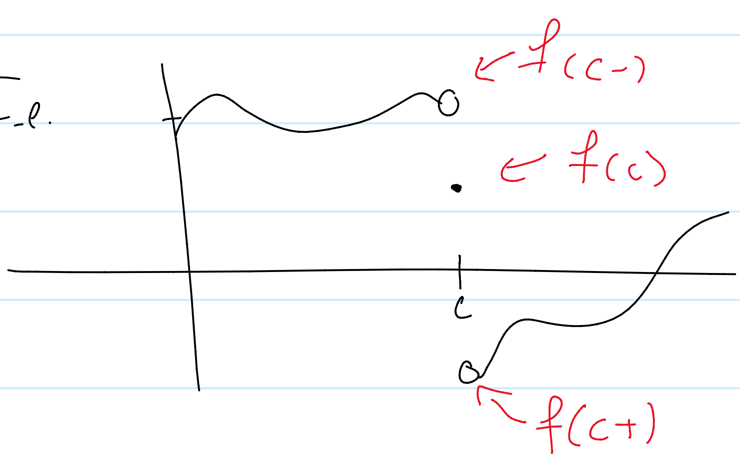
Then

$$\lim_{N\to\infty} S_N(x_0) = \begin{cases} f(x_0) & f \text{ is continuous at } x_0 \\ \dfrac{f(x_0+) + f(x_0-)}{2} & f \text{ is discontinuous at } x_0 \end{cases}$$

Here $f(c+) = \lim_{x\to c^+} f(x)$

$f(c-) = \lim_{x\to c^-} f(x)$.

I.e.



1 $\quad$ 0 $\leftarrow f(c+)$

Take away: If f and f' are piecewise continuous, f is $2L$-periodic function then f has a Fourier series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)$.

For now on f and f' will be piecewise continuous by assumption

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Question:

Let

$$f(x) = \begin{cases} -10 & -20 < x < -15 \\ x & -15 < x < -4\pi \\ x^2 & -4\pi < x < -2 \\ e^x & -2 < x < 10 \\ \sin(x) & 10 < x < 20 \end{cases}$$

Let

$$a_0 = \frac{1}{20}\int_{-20}^{20} f(x)\,dx$$

$$a_n = \frac{1}{20}\int_{-20}^{20} f(x) \cos\left(\frac{n\pi}{20}x\right)dx$$

$$b_n = \frac{1}{20}\int_{-20}^{20} f(x) \sin\left(\frac{n\pi}{20}x\right) dx$$

and

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi}{20}x\right) + b_n \sin\left(\frac{n\pi}{20}x\right)$$

What is

$\lim_{N\to\infty} S_N(x)$? for $x \in [-20, 20]$

$= f(x)$ when

$x \notin \{-20, -15,$
$-4\pi, -2, 10, 20\}$

$$\lim_{N\to\infty} S_N(x) = \begin{cases} \frac{1}{2}\sin(20) \cdot 5 & x = -20, 20 \\ \frac{-25}{2} & x = -15 \\ \frac{-4\pi + (4\pi)^2}{2} & x = -4\pi \\ 2 + \frac{1}{2}e^{-2} & x = -2 \\ \frac{1}{2}(\sin(10) + e^{10}) & x = 10 \end{cases}$$

Last time (on Monday) we approximated a piecewise constant approximated a piecewise constant function. We encountered a strange phenomenon in that case.

The approximation jumps up around a point of discontinuity.

This is typical. It may not always happen, but it happens a lot.

It's known as Gibb's phenomenon.

Properties of even and odd functions.

f is even if
$$f(x) = f(-x)$$
eg: x^2, x^4, $\cos(nx)$, constants

f is odd if
$$f(x) = -f(-x)$$
eg: x, x^3, x^{odd}, $\sin(cx)$

Properties:

1) The sum, difference, product and quotient of 2 even functions is even.
2) The sum and difference of 2 odd functions is odd. Their product and quotients are

Their product and quotients are even.

3) The sum or difference of an even and odd function is neither even or odd (in general). The product or quotient of an even and odd function is odd.

4)
$$\int_{-L}^{L} f(x)\,dx = \begin{cases} 2\int_0^L f(x)\,dx & f \text{ even} \\ 0 & f \text{ odd.} \end{cases}$$

More properties for even functions f, a $2L$-periodic function..

Note f has a Fourier series

$$f(x) = \frac{a_0}{2} + \sum^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right).$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right).$$

What are the coefficients?

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{2}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\cos\left(\frac{n\pi}{L}x\right)}_{\text{even function}}dx = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\sin\left(\frac{n\pi}{L}x\right)}_{\text{odd function}}dx = 0$$

Thus if f is even then

$$\underbrace{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)}_{\text{a cosine series}}$$

For an odd function f, again $2L$-periodic.

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = 0$$

odd functions

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dX = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

Where did we see

$$\frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$ before?

When we solved the heat equation:

$$\begin{cases} u_t = \alpha^2 u_{xx} \end{cases}$$

$$\begin{cases} u_t = \alpha u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\alpha^2 n^2 \pi^2 t/L^2} \sin\left(\frac{n\pi}{L} x\right)$$

where $c_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right).$

Ex) $f(x) = \begin{cases} 0 & x = \pm 2 \\ x & -2 < x < 2 \end{cases}$

make f periodic, $f(x) = f(x+4)$

Note f is an odd function and so

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{2}\right) x$$

$$b_n = \frac{1}{2}\int^2 x \sin\left(\frac{n\pi}{2} x\right) dx$$

$$b_n = \frac{1}{2}\int_{-2} x \sin\left(\frac{n\pi}{2} x\right) dx$$

$$= \int_0^2 x \sin\left(\frac{n\pi}{2} x\right) dx$$

$$= \left(\frac{2}{n\pi}\right)^2 \left[\sin\left(\frac{n\pi}{2} x\right) - \frac{n\pi x}{2} \cos\left(\frac{n\pi}{2} x\right) \right]_0^2$$

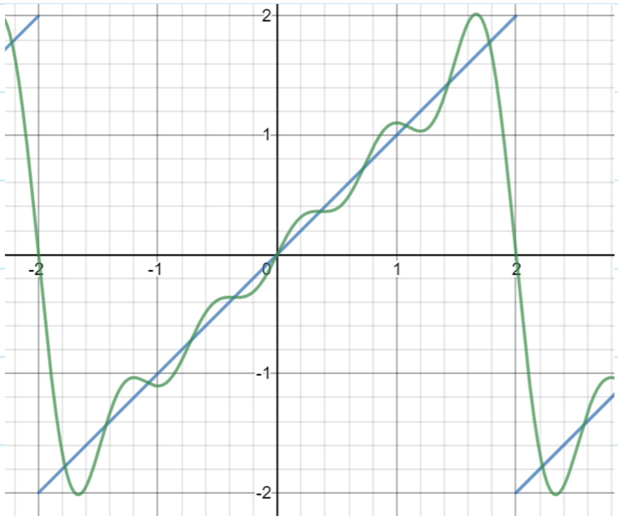
$$= \frac{4}{\pi^2 n^2} \left(- n\pi \cos(n\pi x) - 0 \right)$$

$$= \frac{4}{\pi n} (-1)^{n+1}$$

Thus

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi}{2} x\right).$$

Here is $\frac{4}{\pi} \sum_{n=1}^{5} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi}{2} x\right)$



Now lets recall that

$$g(x) = |x| \quad -2 < x < 2$$

and $g(x) = g(x+4)$ has

Fourier series

$$g(x) = 1 - \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos\left(\frac{(2n-1)\pi}{2} x\right)$$

The functions $|x|$ and x are

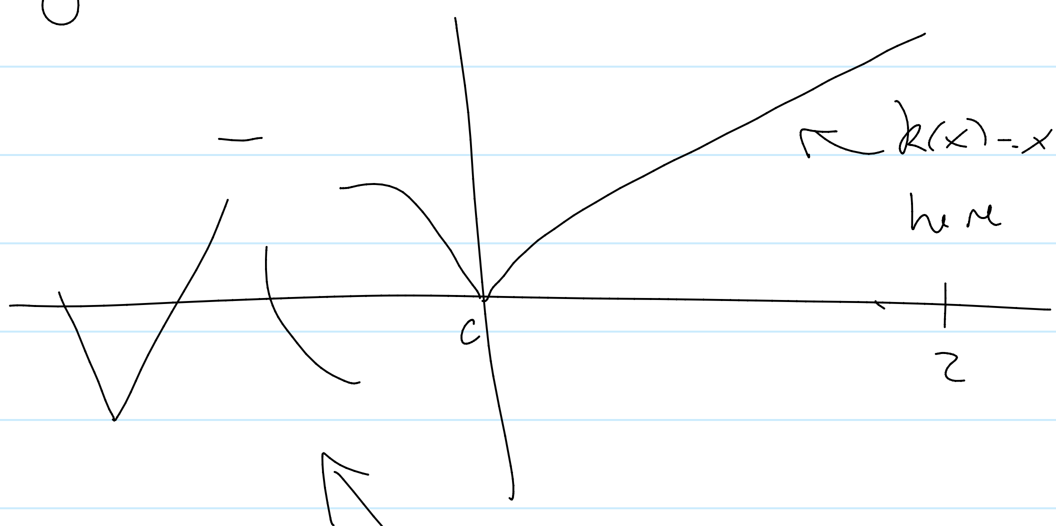
the same for $x \in (0, 2)$

and so on that interval

$(0,2)$ there are $\underline{2}$

(0, 2) There are $\leq$ different ways to write a Fourier series.

In fact there are infinitely many ways. Consider $K(x)$



literally any function on $(-2, 0)$.

will give a different Fourier series.

But on $(0, 2)$ the Fourier series of $K(x)$ will approximate x.