Week 8, Heat Equation Review

Monday, August 12, 2019 6:39 AM

Lets recall how to solve the various heat equations:

Heat equation of a cylindrical rod where the ends are kept at constant temps.

$$\begin{cases} u_t = \alpha^2 u_{xx} \\ \left.\begin{aligned} u(0,t) &= T_1 \\ u(L,t) &= T_2 \end{aligned}\right\} t>0 \\ u(x,0) = f(x) \end{cases}$$

Then

$$u(x,t) = T_1 + \frac{(T_2 - T_1)x}{L} + \sum_{n=1}^{\infty} c_n u_n(x,t)$$

where

$$c_n = \frac{2}{L}\int_0^L \left(f(x) - \left(T_1 + \frac{(T_2-T_1)x}{L}\right)\right)\sin\left(\frac{n\pi}{L}x\right)dx$$

$$c_n = \frac{2}{L}\int_0^L \left[f(x) - \left(T_1 + \frac{(T_2 - T_1)x}{L}\right)\right] \sin\left(\frac{n\pi}{L}x\right) dx$$

and

$$u_n(x,t) = e^{-\alpha^2 n^2 \pi^2 t / L^2} \sin\left(\frac{n\pi}{L}x\right)$$

Recall as $t \to \infty$

$$u(x,t) \to T_1 + \frac{(T_2 - T_1)x}{L}.$$

Heat equation on a cylindrical rod where the ends are insulated:

$$\begin{cases} u_t = \alpha^2 u_{xx} \\ u_x(0,t) = u_x(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

Then

$$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n u_n(x,t)$$

where $$C_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

$$C_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx$$

$$u_n(x,t) = e^{-\alpha^2 n^2 \pi^2 t/L^2} \cos\left(\frac{n\pi}{2}x\right)$$

As $t \to \infty$

$$u(x,t) \longrightarrow \frac{1}{L}\int_0^L f(x)\,dx .$$

Eg)

$$\begin{cases} u_t = u_{xx} \\ u_x(0,t) = u_x(40,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

where $$f(x) = \begin{cases} 10 & x < 10,\ x > 30 \\ 20 & 10 \le x \le 30 \end{cases}$$

Here $L = 40$, $\alpha^2 = 1$ and

what . . ?

$L = 10,\ \alpha = 1$ and

what are c_n?

$$c_n = \frac{2}{40}\int_0^{40} f(x)\cos\left(\frac{n\pi}{40}x\right)dx$$

$$= \frac{1}{2}\int_0^{10}\cos\left(\frac{n\pi}{40}x\right)dx + \int_{10}^{30}\cos\left(\frac{n\pi}{40}x\right)dx + \frac{1}{2}\int_{30}^{40}\cos\left(\frac{n\pi}{40}x\right)dx$$

$$= \frac{40}{n\pi}\left[\frac{1}{2}\sin\left(\frac{n\pi}{40}x\right)\Big|_0^{10} + \sin\left(\frac{n\pi}{40}x\right)\Big|_{10}^{30} + \frac{1}{2}\sin\left(\frac{n\pi}{40}x\right)\Big|_{30}^{40}\right]$$

$$= \frac{40}{n\pi}\left[\frac{1}{2}\sin\left(\frac{n\pi}{4}\right) + \sin\left(\frac{3n\pi}{4}\right) - \sin\left(\frac{n\pi}{4}\right) - \frac{1}{2}\sin\left(\frac{3n\pi}{4}\right)\right]$$

$$= \underbrace{\frac{20}{n\pi}\left[\sin\left(\frac{3n\pi}{4}\right) - \sin\left(\frac{n\pi}{4}\right)\right]}$$

$$\begin{cases} (-1)^{k+1}\, 2/n\pi & n = 2+4k \\ 0 & \text{else} \end{cases}$$

and

$$= \int^{40}$$

and

$$c_0 = \frac{2}{40}\int_0^{40} f(x)\,dx = \frac{2}{40}\left[20\times 10 + 20\times 20\right]$$

$$= 30$$

Thus

$$f(x) = 15 + \frac{40}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2+4k)} \cos\left(\frac{(2+4k)\pi}{4} x\right)$$