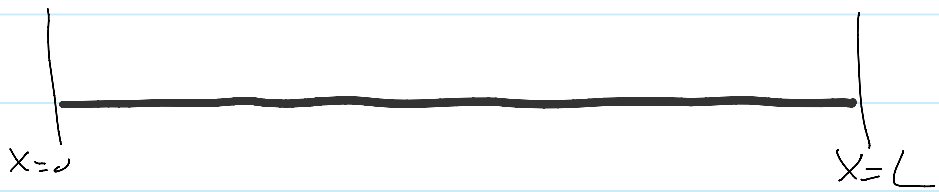
Week 8, Wave Equation

Monday, August 12, 2019 7:10 AM

Suppose we have an elastic string of length L stretched between two supports at the same horizontal level:



If the string is plucked so that it starts moving then (neglecting damping and air resistence) the displacement, say $u(x,t)$, satisfies:

$$u_{tt} = a^2 u_{xx}$$

where $a^2 = T/\rho$ = tension/density.

The ends are kept fixed and so some boundary conditions are

$$u(0,t) = u(L,t) = 0$$

We specify its initial position

$$u(x,0) = f(x)$$

and the strings initial velocity

$$u_t(x,0) = g(x)$$

and we need

$$f(0) = f(L) = g(0) = g(L) = 0$$

Generalizations:

$u_{tt} = a^2(u_{xx} + u_{yy})$ the 2d wave equation, on the head of a drum

$u_{tt} = a^2(u_{xx} + u_{yy} + u_{zz})$ 3d wave equation, for seismic waves on a uniformly dense planet

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Note in order to solve

$$\begin{cases} u_{tt} = a^2 u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

we can find $u^{(1)}$ and $u^{(2)}$

which solve

$$\begin{cases} u^{(1)}_{tt} = a^2 u^{(1)}_{xx} \\ u^{(1)}(0,t) = u^{(1)}(L,t) = 0 \\ u^{(1)}(x,0) = f(x) \\ u^{(1)}_t(x,0) = 0 \end{cases}$$

and

$$\begin{cases} u^{(2)}_{tt} = a^2 u^{(2)}_{xx} \\ u^{(2)}(0,t) = u^{(2)}(L,t) = 0 \\ u^{(2)}(x,0) = 0 \\ u^{(2)}_t(x,0) = g(x) \end{cases}$$

and then

$$u(x,t) = u^{(1)} + u^{(2)}.$$

Let's start with

$$\begin{cases} u_{tt} = a^2 u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x), \quad u_t(x,0) = 0 \end{cases}$$

Using separation of variables we guess. $u(x,t) = X(x)T(t)$.

Then

$$u_{xx} = X''T$$

$$u_{tt} = XT''$$

so

$$XT'' = a^2 X''T.$$

Hence

$$\frac{X''}{X} = \frac{1}{a^2}\frac{T''}{T} = -\lambda$$

as we have seen several times already.

$$X'' + \lambda X = 0$$

$$T'' + a^2\lambda T = 0$$

What are the boundary conditions?

What are the boundary conditions?

Well $u(0,t) = u(L,t) = 0$

$$\underbrace{u(0,t)}_{X(0)T(t)} = \underbrace{u(L,t)}_{X(L)T(t)} = 0$$

and so $X(0) = X(L) = 0$.

Thus X solves:

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(L) = 0 \end{cases}$$

and hence: eigenval's & eigen fns

λ_n	X_n
$\dfrac{n^2\pi^2}{L^2}$	$\sin\left(\dfrac{n\pi}{L}x\right)$.

Using these λ_n's we have

$$T'' + \frac{a^2 n^2 \pi^2}{L^2} T = 0$$

and so

$$T(t) = k \cos\left(\underline{\pi n a}\, t\right) + k \sin\left(\underline{\pi n a}\, t\right)$$

$$T_n(t) = k_1 \cos\left(\frac{\pi n a}{L} t\right) + k_2 \sin\left(\frac{\pi n a}{L} t\right)$$

Recall $u_t(x,0) = 0$

$$\| \\ X(x)\, T'(0)$$

hence $T'(0) = 0$.

Using this initial condition

$$T_n(t) = k_1 \cos\left(\frac{\pi n a}{L} t\right)$$

Now we <u>add</u> together all these solutions to say

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L} x\right) \cos\left(\frac{n\pi}{L} a t\right)$$

changed what we called the constants.

Using the initial conditions

$$u(x,0) = f(x)$$

$$u(x,0) = f(x)$$
$$\| $$
$$\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right)$$

hence $$c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

Eg)
$$\begin{cases} u_{tt} = 16u_{xx} \\ u(0,t) = u(40,t) = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = 0 \end{cases}$$

Where
$$f(x) = \begin{cases} 4x & x \le 10 \\ 80-4x & 10 < x \le 20 \\ 0 & 20 \le x \le 40 \end{cases}$$

$$c_n = \frac{2}{40}\int_0^{40} f(x)\sin\left(\frac{n\pi}{40}x\right)dx$$

$$= \frac{1}{20}\left[\int^{10} 4x\sin\left(\frac{n\pi}{40}x\right)dx + \int^{20}(80-4x)\sin\left(\frac{n\pi}{40}x\right)dx\right]$$

$$= \frac{1}{20}\left[\int_0 4x\sin\left(\frac{n\pi}{40}x\right)dx + \int (80-4x)\sin\left(\frac{n\pi}{40}x\right)dx\right]$$

$$= \frac{1}{20}\left[\frac{1600}{n^2\pi^2}\left(4\sin\left(\frac{n\pi}{4}\right) - n\pi\cos\left(\frac{n\pi}{4}\right)\right)\right]$$

$$-\frac{1}{20}\left[\frac{1600}{n^2\pi^2}\left(4\sin\left(\frac{\pi n}{2}\right) - \pi n\cos\left(\frac{n\pi}{4}\right) - 4\sin\left(\frac{n\pi}{4}\right)\right)\right]$$

$$= \frac{80}{n^2\pi^2}\left[8\sin\left(\frac{n\pi}{4}\right) - 4\sin\left(\frac{\pi n}{2}\right)\right]$$

So

$$u(x,t) = \frac{320}{\pi^2}\sum_{n=1}^{\infty}\frac{2\sin\left(\frac{n\pi}{4}\right) - \sin\left(\frac{\pi n}{2}\right)}{n^2}\sin\left(\frac{n\pi}{40}x\right)\cos\left(\frac{n\pi}{10}t\right)$$

Comments:

For fixed n:

$\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi}{L}at\right)$ is

periodic in t with period

$$\frac{2L}{n a}.$$

For $n = 1, 2, \cdots$ the quantities $\frac{n\pi a}{L}$ are the natural frequencies of the spring.

Ie the string will freely vibrate at these frequencies.

The displacement patterns are called the natural modes of the string. Ie the $\sin\left(\frac{n\pi}{L} x\right)$ are the natural modes. Their periods $(2L$ called the wavelengths.

What happens when we have this wave equation!

wave in 1D wave equation:

$$\begin{cases} u_{tt} = a^2 u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = 0 \\ u_t(x,0) = g(x). \end{cases}$$

Well we again use separation of variables:

$$u(x,t) = XT$$

$$u_{tt} = XT'' = a^2 X''T = a^2 u_{xx}$$

$$\Rightarrow \frac{T''}{a^2 T} = \frac{X''}{X} = -\lambda$$

So

$$X'' + \lambda X = 0$$

What are the boundary conditions?

$$0 = u(0,t) = X(0)T(t) \Rightarrow X(0) = 0$$

$$0 = u(L,t) = X(L)T(t) \Rightarrow X(L) = 0$$

So

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(L) = 0 \end{cases}$$

which has eigenvalues

λ_n	X_n
$\frac{n^2\pi^2}{L^2}$	$\sin\left(\frac{n\pi}{L}x\right)$.

For these λ_n we solve the ODE for T.

$$T'' + \frac{a^2 n^2 \pi^2}{L^2} T = 0$$

$$T = k_1 \cos\left(\frac{an\pi}{L} t\right) + k_2 \sin\left(\frac{an\pi}{L} t\right)$$

What initial conditions are there?

Well $0 = u(x,0) = X(x)\,T(0)$

$\Rightarrow$ $T(0)$ is 0

Hence $T(t) = k \sin\left(\frac{an\pi}{L} t\right)$

Some unknown constant

Adding together all solutions:

Adding together all solutions:

$$u(x,t) = \sum_{n=1}^{\infty} k_n \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{an\pi}{L}t\right)$$

changed the constant

What are k_n?

Well

$$g(x) = u_t(x,0) = \frac{\partial}{\partial t} \sum_{n=1}^{\infty} k_n \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{an\pi}{L}t\right) \Bigg|_{t=0}$$

$$= \sum_{n=1}^{\infty} \frac{an\pi}{L} k_n \sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{an\pi}{L}t\right) \Bigg|_{t=0}$$

$$= \sum_{n=1}^{\infty} \underbrace{\frac{an\pi}{L} k_n}_{\text{Fourier coefficients for the sine series of } g(x).} \sin\left(\frac{n\pi}{L}x\right)$$

Thus

$$\frac{an\pi}{L} k_n = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

$$\Rightarrow \quad k_n = \frac{2}{an\pi} \int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

What about:

$$\begin{cases} u_{tt} = a^2 u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

is given by adding together the two solutions:

$$u(x,t) = \sum_{n=1}^{\infty} \left(c_n \cos\left(\frac{an\pi}{L} t\right) + k_n \sin\left(\frac{an\pi}{L} t\right) \right) \sin\left(\frac{n\pi}{L} x\right)$$

where $c_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx$

$$k_n = \frac{2}{an\pi} \int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right) dx$$