Week 8/9 Laplace's Equation

Wednesday, August 14, 2019 7:22 AM

Laplace's equation is

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{2-d} \\ u_{xx} + u_{yy} + u_{zz} = 0 & \text{3-d} \\ \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} u = 0 & \text{in n-d} \end{cases}$$

We'll write $\boxed{\Delta u = 0 \quad \text{in general}}$

This is sometimes called the potential equation. The reason why has to do with potential energy in physics.

For example, the gravitational potential energy is a constant multiple of

$$U(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{1}{r}$$

and

$$\frac{\partial^2}{\partial x^2} U = \frac{2x^2 - y^2 - z^2}{r^{5/2}}$$

$$\frac{\partial^2}{\partial y^2} U = \frac{2y^2 - x^2 - z^2}{r^{5/2}}$$

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$$\frac{\partial^2}{\partial z^2} u = \frac{2z^2 - x^2 - y^2}{r^{5/2}}$$

Adding these up gives

$$\frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u + \frac{\partial^2}{\partial z^2} u$$

$$= \frac{2(x^2+y^2+z^2) - 2(x^2+y^2+z^2)}{r^{5/2}} = 0$$

So the gravitational potential energy solves Laplace's equation in 3-d.

With the heat equation & wave equation we specified boundary values for the x variable as t evolved

$$u(0,t) = u(L,t) = 0.$$

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and initial conditions for t:

$$u(x,0) = f(x), \quad u_t(x,0) = g(x).$$

Now

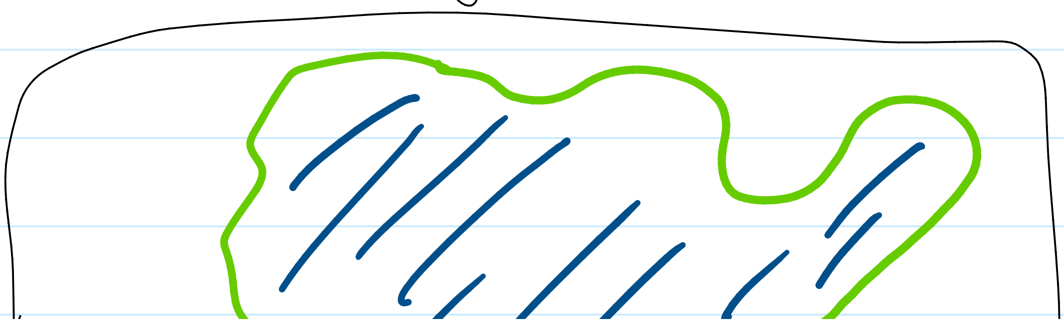
$$\Delta u = 0$$

has no dependence on time.

How many boundary conditions do we need?

Well there are two different sets of problems we can try to solve:

1) We have a region D in space and we specify what happens on the boundary of D, the boundary is denoted ∂D.



The green is ∂D
the blue is D.

Specifying what happens to $u(x,y)$ on ∂D is called the Dirichlet problem

2) We can specify what happens to the derivative of u on ∂D. This is called the Neumann problem.

In order to solve the Laplace equation on a region D and guarantee that there is only 1 solution (ie existence & uniqueness) we need conditions on ∂D. The full scope of this is beyond full scope of this is beyond what we could do in several months of work, so we'll focus on rectangles and circles, and the Dirichlet problem

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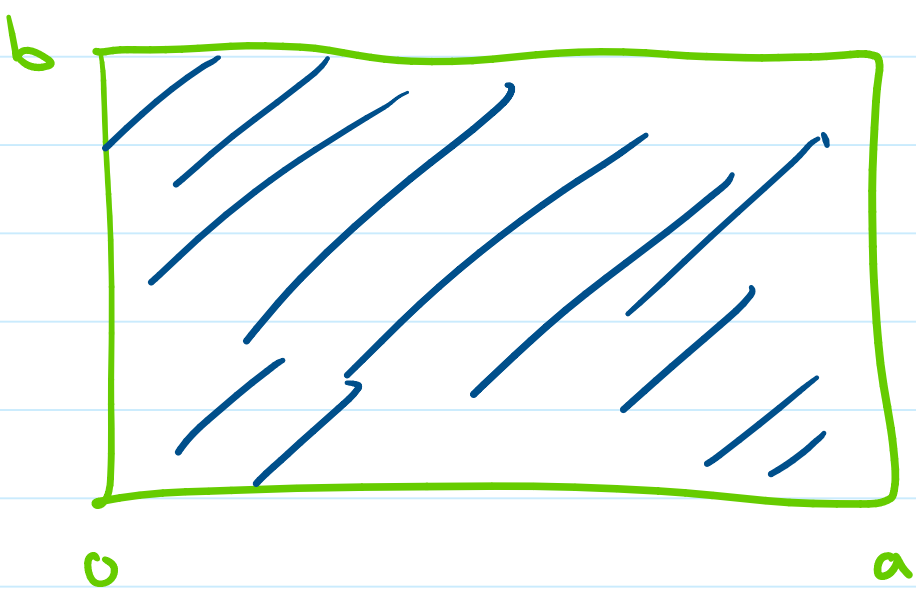
Dirichlet problem on rectangle

The domain D is

$$D = \{(x,y) : 0 \leq x \leq a,\ 0 \leq y \leq b\}$$



What is ∂D?

$$\partial D = \left\{ (x,y) : \begin{array}{ll} x=0,a \text{ and } 0 \le y \le b \\ \text{or } y=0,b \text{ and } 0 \le x \le a \end{array} \right\}$$

in green.

So the Dirichlet problem (in general) is

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ \left.\begin{array}{l} u(0,y) = f_L(y) \\ u(a,y) = f_R(y) \end{array}\right\} 0 \le y \le b \\ \left.\begin{array}{l} u(x,0) = g_L(x) \\ u(x,a) = g_R(x) \end{array}\right\} 0 < x < a \end{cases}$$

We will solve the problem when only $f_R(y)$ is nonzero

Ie
$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(0,y) = 0 \qquad a \le y \le b \\ u(a,y) = f(y) \\ u(x,0) = u(x,b) = 0, \quad 0 < x < a \end{cases}$$

We suppose
$$u = XY \text{, as usual.}$$

Then
$$X''Y + XY'' = 0$$
divide by XY to get

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$
$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

Then
$$X'' - \lambda X = 0$$
$$Y'' + \lambda Y = 0$$

What are the boundary conditions?

Well

$$u(x,0) = X(x)\,Y(0) = 0$$
$$u(x,b) = X(x)\,Y(b) = 0$$

so
$$\begin{cases} Y'' + \lambda Y = 0 \\ Y(0) = 0 \\ Y(b) = 0 \end{cases}$$

hence there are eigenvalues & eigenfunctions:

λ_n	Y_n
$\frac{n^2\pi^2}{b^2}$	$\sin\left(\frac{n\pi}{b} y\right)$.

What boundary conditions for X?

$$u(0,y) = 0 = X(0)\,Y(y).$$

So
$$\begin{cases} X_n'' - \frac{n^2\pi^2}{b^2} X = 0 \\ X(0) = 0 \end{cases}$$

$$\begin{cases} \quad " \qquad b^{\leftarrow} \quad " \\ X(0) = 0 \end{cases}$$

The solution to the ODE:

$$X'' - \mu^2 X = 0$$ are

$$X = c_1 e^{\mu x} + c_2 e^{-\mu x}$$

However we can also write this as

$$X(x) = k_1 \sinh(\mu x) + k_2 \cosh(\mu x)$$

where $\sinh(z) = \dfrac{e^{z} - e^{-z}}{2}$

$$\cosh(z) = \frac{e^{z} + e^{-z}}{2}$$

$$\cosh'(z) = \sinh(z)$$
$$\sinh'(z) = \cosh(z).$$

So but

$$\cosh(0) = 1 \quad , \sinh(0) = 0$$

So
$$X_n = c_n \sinh\left(\frac{n\pi}{b}x\right)$$

Adding all the solutions gives:

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{b}y\right) \sinh\left(\frac{n\pi}{b}x\right)$$

Now
$$u(a,y) = \underbrace{\sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi}{b}a\right) \sin\left(\frac{n\pi}{b}y\right)}$$
$$= f(y)$$

Thus the Fourier sine series has coefficients $c_n \sinh\left(\frac{n\pi}{b}a\right)$

Hence
$$c_n \sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b}\int_0^b f(y) \sin\left(\frac{n\pi}{b}y\right) dy$$

$C_n = 2 \int^b$

$$c_n = \frac{2}{b\sinh\left(\frac{n\pi a}{b}\right)} \int_0^b f(y)\sin\left(\frac{n\pi}{b}y\right)dy$$

Thus

$$u(x,y) = \sum_{n=1}^{\infty} \left[\frac{2}{b}\int_0^b f(y)\sin\left(\frac{n\pi y}{b}\right)dy\right] \frac{\sinh\left(\frac{n\pi x}{b}\right)}{\sinh\left(\frac{n\pi a}{b}\right)} \sin\left(\frac{n\pi}{b}y\right)$$

How does this behave for large n?

For large n, $x \neq 0$

$$\sinh\left(\frac{n\pi x}{b}\right) = \frac{e^{n\pi x/b} - e^{-n\pi x/b}}{2}$$

$$\approx \frac{e^{n\pi x/b} - 0}{2} = \frac{e^{n\pi x/b}}{2}$$

Similarly

$$\frac{\sinh\left(\frac{n\pi x}{b}\right)}{\sinh\left(\frac{n\pi a}{b}\right)} = \frac{\frac{1}{2}e^{n\pi x/b}}{\frac{1}{2}e^{n\pi a/b}}$$

$$= \exp\left[\frac{-n\pi\,(a-x)}{b}\right]$$

So as n gets larger

$\exp\left[\frac{-n\pi\,(a-x)}{b}\right]$ gets smaller

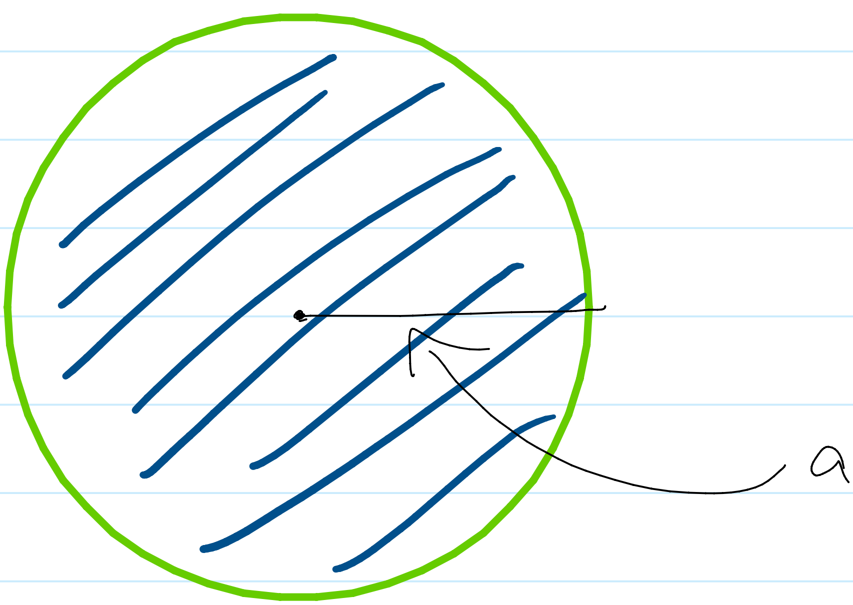
unless $(a-x)$ is really small.

Laplace's equation on a disk:

Consider a disk of radius $a>0$.

$D = \{(x,y) : x^2+y^2 \le a^2\}$

$\partial D = \{(x,y) : x^2+y^2 = a^2\}$



Laplaces equation on the disk can be written in polar coordinates

$$u(x,y) = u(r,\theta)$$

where

$$x^2 + y^2 = r^2$$

$$r\cos\theta = x \quad , \quad r\sin\theta = y.$$

The chain rule can be used to show Laplace's equation is

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

The boundary conditions are

$$u(a,\theta) = f(\theta)$$

How to solve

$$\begin{cases} u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \\ u(a, \theta) = f(\theta) \end{cases}$$

$$u(r,\theta) = R(r)\, \Theta(\theta)$$

So the PDE becomes

$$R'' \Theta + \frac{1}{r} R' \Theta + \frac{1}{r^2} R \Theta'' = 0$$

Rearranging gives

$$\frac{r^2 R'' + r R'}{R} = \frac{-\Theta''}{\Theta} = \lambda$$

Hence:

$$r^2 R'' + r R' - \lambda R = 0$$

$$\Theta'' + \lambda \Theta = 0$$

Since $\Theta(\theta + 2\pi) = \Theta(\theta)$

we need

we need

$$\lambda \geq 0.$$

So we write $\lambda = \mu^2$.

Cases

$\mu = 0$ $\qquad \Theta'' = 0$

$$\Theta(\theta) = c_1 + c_2 x$$

Periodic tells us $c_2 = 0$.

Thus
$$\boxed{\begin{array}{ll} \mu = 0 & \Theta \text{ is constant} \\ \lambda = 0 & \end{array}}$$

Otherwise

$\mu \neq 0$ so $\Theta(\theta) = c_1 \cos \mu\theta + c_2 \sin \mu\theta$

in order for $\Theta(\theta + 2\pi) = \Theta(\theta)$

we need $\mu = 1, 2, 3, \ldots$

$$\boxed{\begin{array}{ll} \mu = n, & \Theta = c_1 \cos(n\theta) + c_2 \sin(n\theta) \\ \lambda = n^2 & \end{array}}$$

What are R in these cases?

$\mu = 0$

$$\underbrace{r^2 R'' + r R'} = 0 \quad , \text{ Say } \quad Q(r) = R'(r)$$

$$r^2 Q' + r Q = 0$$

$$\frac{dQ}{Q} = -\frac{dr}{r} \Rightarrow \log Q = -\log r$$

$$\Rightarrow Q = \frac{c_1}{r} = R'$$

Hence

$$R(r) = k_1 + k_2 \ln(r)$$

If $k_2 \neq 0$ then

$R(r) \to \pm\infty$ as

$r \to 0$.

So

$$\boxed{\begin{array}{lll} \mu = 0 & \Theta & \text{is constant} \\ \lambda = 0 & & \\ & R & \text{is constant} \end{array}}$$

So

$$\boxed{\begin{array}{lll} \mu = 0 & \Theta & \text{is constant} \\ \lambda = 0 & & \\ & R & \text{is constant} \end{array}}$$

What about when $\lambda = n^2$

Then

$$r^2 R'' + r R' - n^2 R = 0$$

What functions relate rR' to R?

Well r^k would work

$$r (r^k)' = k r^k .$$

So what if

$$R(r) = r^\alpha$$

Then

$$r^2 R'' + r R' - n^2 R = 0$$

$$= \alpha(\alpha-1) r^{\alpha} + \alpha r^{\alpha} - n^2 r^{\alpha} = 0$$

$$= (\alpha^2 - n^2) r^{\alpha} = 0 \quad \Rightarrow \quad \alpha^2 = n^2$$

Hence

$$R(r) = k_1 r^n + k_2 r^{-n}.$$

But if $r \to 0$ and $k_2 \neq 0$
then $R(r) \to \pm\infty$.
So $k_2 = 0$

$$\boxed{\begin{array}{ll} \mu = n & \Theta = c_1 \cos(n\theta) + c_2 \sin(n\theta) \\ \lambda = n^2 & R(r) = k_1 r^n. \end{array}}$$

Taking the infinite sum gives

$$u(r,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^n \left(c_n \cos(n\theta) + k_n \sin(n\theta)\right)$$

Using $u(a, \theta) = f(\theta)$ gives

$$f(\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} a^n c_n \cos(n\theta) + a^n k_n \sin(n\theta)$$

so $a^n c_n$ and $a^n k_n$ are the Fourier coefficients for f.

$$c_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta$$

$$a^n c_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(n\theta) f(\theta)\, d\theta$$

$$a^n k_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(n\theta) f(\theta)\, d\theta$$

Hence

$$c_n = \frac{a^{-n}}{\pi}\int_{-\pi}^{\pi} \cos(n\theta) f(\theta)\, d\theta$$

$$k_n = \frac{a^{-n}}{\pi}\int^{\pi}$$

$$k_n = \frac{a^{-n}}{\pi} \int_{-\pi}^{\pi} \sin(n\theta)\, f(\theta)\, d\theta.$$