

A MULTI-TYPE RAY-KNIGHT THEOREM

DAVID CLANCY

ABSTRACT. The Ray-Knight theorem describes the path behavior in the spacial variable of the local time of a Brownian motion stopped at appropriate times and the resulting process is a continuous state branching process. We show that under appropriate assumptions, a multi-type branching process with immigration can be obtained as the local times of a system of stochastic processes. These assumptions are easily seen to be met in the Brownian setting, and more generally an α -stable setting for $\alpha \in (1, 2]$. Using these processes, we also extend the Gorin-Shkolnikov identity [18] to certain random fields.

1. INTRODUCTION

The well known Ray-Knight theorems describe the behavior in the spacial variable of the local time of a Brownian motion at certain stopping times T . They originate with the works of Ray [31] and Knight [22]. These results have been extended in numerous aspects to various other processes and contexts. Using the interpretation of the Ray-Knight theorems implicit in works of Duquesne, Le Gall and Le Jan in [27, 26, 15, 14], we can describe the Ray-Knight theorems as obtaining many continuous state branching processes with immigration in terms of the local time of a “height” process (for more information on these processes see Section 3). Their work relies on random trees and random forests being encoded by a height function and a branching process structure. In this paper we attempt to answer the question: Can you obtain multi-type continuous state branching processes as the local times of a solution to some system of stochastic equations? We do this by encoding multi-type forests by a collection of height processes.

On a similar note, the work of Chaumont and Liu [8] encode multi-type forests (without immigration) in a manner quite similar to ours and use that encoding to extend the ballot theorem. The authors of that work consider distributional limits but avoid process-level limits. Miermont [29] also encodes multi-type branching forests using a single height process and proves continuum limit theorems under certain assumptions including finite variance and criticality. The author also shows, under an exponential moment assumption, a conditioned version of the result, in which the limiting object is a (constant multiple of a) Brownian excursion. See also Berzunza Ojeda’s work in [4] for similar results outside of a Brownian regime and de Raphélis’s work in [10] for an extension to infinitely many types.

Let’s briefly look at the work of Abraham and Mazliak [1]. Expanding on previous works on the branching properties of Brownian paths, they show that a weak solution to the stochastic differential equation

$$dZ_v = 2\sqrt{Z_v}dW_v + \Delta'(v)dv, \quad Z_0 = 0$$

for unbounded, C^1 and strictly increasing function Δ can be obtained by looking at the local time of the process

$$H_t = \beta_t + \Delta^{-1}(\ell_t)$$

where β is a reflected Brownian motion and ℓ_t is its local time at time t and level 0. The authors give a nice tree picture, which is also described in [30]. For another description of this result, see

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[28]. We can rewrite the process H in the following way

$$(1) \quad \begin{cases} H_t = \beta_t + J_t \\ J_t = \int_0^t \frac{1}{\Delta'(J_s)} d\ell_s \end{cases} .$$

The characterization above is what we wish to generalize to systems of stochastic processes.

1.1. Statement of Results. We will consider systems based on N independent Lévy processes $\mathbf{X}^1, \dots, \mathbf{X}^N$. In order for our proof techniques to follow through, we must make a few technical assumptions. We assume that for each $j \in [N] := \{1, \dots, N\}$

$$\mathbb{E} \left[\exp \left\{ -\lambda \mathbf{X}_1^j \right\} \right] = \exp \{ \psi_j(\lambda) \}$$

where

$$(2) \quad \begin{aligned} \psi_j(\lambda) &= \alpha_j \lambda + \beta_j \lambda^2 + \int_{(0, \infty)} (e^{-\lambda r} - 1 - \lambda r 1_{[r < 1]}) \pi_j(dr) \\ \int_1^\infty \frac{1}{\psi_j(u)} du &< \infty \end{aligned}$$

with $\alpha_j, \beta_j \geq 0$ and π_j is a Radon measure on $(0, \infty)$ with $\int_{(0, \infty)} (r \wedge r^2) \pi_j(dr) < \infty$. The conditions above are the technical conditions of [15] which guarantee certain nice properties of height processes, which will be discussed later in Section 3.2.

We also must make an assumption that these Lévy processes can arise under the *same scaling* which we now describe. We say that a probability measure on \mathbb{N} is (sub)critical if $\sum_{k \geq 0} k \mu(k) \leq 1$. We assume that there exists sequences of (sub)critical probability measures $(\mu^{j,p}; j \in [N], p \geq 1)$ on $\mathbb{N}_0 = \{0, 1, \dots\}$ with generating functions $g^{j,p}$. Recursively define $g_n^{j,p}$ by $g_n^{j,p} = g_{n-1}^{j,p} \circ g^{j,p}$ with $g_0^{j,p} = id$. We also let $(\xi_\ell^{j,p}; \ell \geq 0)$ be independent random variables where the law of $\xi_\ell^{j,p}$ is $\mu^{j,p}$ for each ℓ . We make the assumption that there exists an increasing sequence $(\gamma_p; p \geq 1)$ of non-negative integers such that for each $j \in [N]$

$$(3) \quad \begin{aligned} \frac{1}{p} \sum_{\ell=1}^{p\gamma_p} (\xi_\ell^{j,p} - 1) &\implies \mathbf{X}_1^j \\ \liminf_{p \rightarrow \infty} g_{[\delta \gamma_p]}^{j,p}(0) &> 0, \quad \forall \delta > 0 \end{aligned}$$

Definition 1. We call a family of Laplace exponents (ψ_j) for $j \in [N]$ **admissible** if they satisfy (2) and can their corresponding Lévy processes can be obtained by (3).

Remark 1.1. The definition of admissible may seem to be quite restrictive; however, Theorem 2.3.2 in [15] implies that $\psi_j(\lambda) = c_j \lambda^\alpha$ for $\alpha \in (1, 2)$ are admissible. Also, the Lindeberg-Feller CLT implies that $\psi_j(\lambda) = \alpha_j \lambda + \beta_j \lambda^2$ are admissible for $\alpha_j \geq 0, \beta_j, c_j > 0$. Moreover, anytime [15, Corollary 2.5.1] applies, we can find a family of admissible ψ_j 's.

As mentioned above, if $(\psi_j; j \in [N])$ are admissible, then there exists a ψ_j -height processes $\mathbf{H}^j = (\mathbf{H}_t^j; t \geq 0)$ for each $j \in [N]$. The value \mathbf{H}_t^j roughly “measures” in a local time sense the size of the set

$$\{s \leq t : \inf_{r \in [s, t]} \mathbf{X}_r^j = \mathbf{X}_{s-}^j\}.$$

More details of this process will be discussed in Section 3.2. Moreover due to integral assumption in (2), there exists a continuous modification of the process \mathbf{H}^j , and from now on we will only consider this modification.

We let $\ell_t^j = (\ell_t^j; t \geq 0)$ denote the local time at time t and level 0 of the process \mathbf{H}^j . It is defined as the L^1 -limit (see [15, Lemma 1.3.2]):

$$(4) \quad \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t 1_{[\mathbf{H}_s^j \leq \varepsilon]} ds = \ell_t^j = - \inf_{s \leq t} \mathbf{X}_s^j.$$

We also let $\tau_x^j = \inf\{t : \ell_t^j > x\}$ be the right-continuous inverse of ℓ^j .

We wish to generalize equation (1) to systems of equations. To do this, we fix $\delta_j > 0$, $x_j \geq 0$ for $j \in [N]$ and $\alpha_{i,j} \geq 0$ for $i \neq j$. We study the following system of stochastic equations

$$(5) \quad \begin{cases} \overleftarrow{\mathbf{H}}_t^j = \mathbf{H}_t^j + \mathbf{J}_t^j \\ \mathbf{J}_t^j = \int_{\tau_{x_j}^j}^{t \vee \tau_{x_j}^j} \left\{ \delta_j + \sum_{i \neq j} \alpha_{i,j} \mathbf{L}_\infty^{\mathbf{J}_s^j}(\overleftarrow{\mathbf{H}}^i) \right\}^{-1} d\ell_s^j \end{cases},$$

where $\mathbf{L}_\infty^v(\overleftarrow{\mathbf{H}}^j)$ is a local time of the process $\overleftarrow{\mathbf{H}}^j$.

More precisely, we prove that existence to the equation above, which is stated as the following theorem:

Theorem 1.1. *Fix any $\delta_j > 0$, $x_j \geq 0$ for $j \in [N]$ and any $\alpha_{i,j} \geq 0$ for $i \neq j \in [N]$. Then for any admissible family of Laplace exponents $(\psi_j)_{j \in [N]}$, and independent ψ_j -height processes \mathbf{H}^j there exists a weak solution to the stochastic equation (5). Moreover, in the solution the processes $\mathbf{L}_\infty^v(\overleftarrow{\mathbf{H}}^j)$ are càdlàg processes in v and, almost surely, for any continuous function g with compact support*

$$(6) \quad \int_0^\infty g(\overleftarrow{\mathbf{H}}_t^j) dt = \int_0^\infty g(v) \mathbf{L}_\infty^v(\overleftarrow{\mathbf{H}}^j) dv$$

From now on, when we discuss solutions to the equation (5) we will implicitly assume that are speaking about the solution given by Theorem 1.1. We can state the following corollary to Theorem 1.1, which is just a simple case where we know a family of ψ_j are admissible.

Corollary 1.2. *Suppose that $\delta_j > 0$, $x_j \geq 0$ for $j \in [N]$ and $\alpha_{i,j} \geq 0$ for all $i \neq j \in [N]$. We suppose $\alpha_{j,j} \leq 0$. Let B_t^j denote independent standard Brownian motions. Then, for any $\beta_j > 0$ there exists a solution to (5) for the height processes defined by*

$$\mathbf{H}_t^j = \frac{1}{\beta_j} \left(\sqrt{2\beta_j} B_t^j + \alpha_{j,j} t - \inf_{s \leq t} \{ \sqrt{2\beta_j} B_s^j + \alpha_{j,j} s \} \right).$$

In order to prove Theorem 1.1, we construct a forest model where the solution arises fairly naturally. The model is discussed in Section 2. The forest model also gives rise to a Ray-Knight type theorem, which requires more information on multi-type continuous state branching processes. In lieu of discussing the full set up here, we discuss the Ray-Knight theorem in the context of Corollary 1.2 and reserve the more general result for Proposition 4.5:

Corollary 1.3. *Suppose the hypothesis of Corollary 1.2 and for each $j \in [N]$ let $\mathbf{Z}_v^j = \mathbf{L}_\infty^v(\overleftarrow{\mathbf{H}}^j)$. Then \mathbf{Z} is a weak solution to the system stochastic differential equations*

$$(7) \quad d\mathbf{Z}_v^j = \sqrt{2\beta_j} \mathbf{Z}_v^j dW_v^j + \left\{ \delta_j + \sum_{i=1}^N \alpha_{i,j} \mathbf{Z}_v^i \right\} dv, \quad \mathbf{Z}_0^j = x_j,$$

where (W^1, \dots, W^N) is an \mathbb{R}^N -valued Brownian motion.

Remark 1.2. We remark that in the Brownian case of Corollary 1.2, the solution $\overleftarrow{\mathbf{H}}^j$ is a semi-martingale with quadratic variation $\langle \overleftarrow{\mathbf{H}}^j \rangle_t = \frac{2t}{\beta_j}$. Hence, if $L_\infty^v(\mathbf{H}^j)$ is the semi-martingale local time then $L_\infty^v(\overleftarrow{\mathbf{H}}^j) = \frac{\beta_j}{2} \mathbf{L}_\infty^v(\overleftarrow{\mathbf{H}}^j)$. Thus there is a similar statement to Corollary 1.3 where we replace \mathbf{L} with L . In this situation, we can also replace the constants in equation (5) in order to have \mathbf{J}_t^j depend on $L_\infty^{\mathbf{J}_s^j}(\overleftarrow{\mathbf{H}}^i)$.

After proving the above results, we extend the result of Abraham and Mazliak [1] in the $N = 1$ Brownian case. Namely, we argue the following Theorem:

Theorem 1.4. *Let F be any strictly increasing càdlàg function diverging towards infinity with $F(0) = 0$, and let $x \geq 0$. Then there exists a semi-martingale $\overleftarrow{\mathbf{H}} = (\overleftarrow{\mathbf{H}}_t; t \geq 0)$ whose local time $\mathbf{Z}_v = \mathbf{L}_\infty^v(\overleftarrow{\mathbf{H}})$ is a weak solution to the stochastic equation*

$$(8) \quad \mathbf{Z}_v = x + 2 \int_0^v \sqrt{\mathbf{Z}_u} dW_u + F(v)$$

for a Brownian motion W .

Under some differentiability conditions on F , stochastic equations in (8) were studied in connection to long-term interest rate models in [11, 12]. See also [16].

1.2. Organization of the Paper. In Section 2 we discuss the forest model and establish the notation that will be used throughout the sequel. This section contains within it many identities that happen in the discrete which will be useful in our later weak convergence arguments. Section 2.3 describes the randomization of the model, where the discrete processes defined on forests become well-known discrete stochastic processes.

In Section 3 we give a brief overview of (multi-type) continuous state branching processes along with their connections to Lévy processes. We do discuss their characterization as affine processes; however, all the results in the sequel refers to them through the useful time-change found in [7]. Section 3.2 also gives a brief overview of the ψ -height processes, under the assumptions discussed in Section 1.1. We do point to references where weaker assumptions are made on the Laplace exponents ψ . Shortly after discussing general multi-type continuous state branching processes, we restrict the types of multi-type branching processes we consider.

Section 4 contains the main weak convergence arguments. We recall several results that follow from the existing literature that will be useful to state explicitly in this paper. Section 4.2 is devoted to proving Theorem 4.4 from which Theorem 1.1 follows from simple observations. We then state the general Ray-Knight theorem with Proposition 4.5. In Section 4.3, we show how Corollary 1.2 follows from Theorem 1.1 and how Corollary 1.3 follows from Proposition 4.5.

In Section 5, we prove Theorem 1.4 and prove various results about solutions to the equation in (8).

In Section 6 we discuss a nice consequence of these theorems. We briefly discuss it here as well. Using random matrix theory, Gorin and Sholnikov [18] have shown a distributional identity of the difference between the area under a Brownian excursion and the integral of the square of its total local time. This was then proved using different techniques by Hariya [20], and extended to reflected Brownian bridges by Gaudreau Lamarre and Shkolnikov in [23]. This was later extended by the author in [9] to include ψ -height processes where ψ satisfies (2). We extend this identity to the solutions $\overleftarrow{\mathbf{H}}^j$ to (5). This is contained in Theorem 6.1.

2. DESCRIPTIONS OF FORESTS

In this section we describe how our discrete forests are constructed, and what processes on the forest we define. A forest, say \mathfrak{f} , will be both rooted and colored by the *colors* or *types* $0, 1, \dots, N$. The type zero vertices will be of a special kind when we randomize the model. For the colors $1, \dots, N$ we will define two separate labelings on the vertices, a breadth-first ordering and a depth-first ordering. Many of the processes we define are described by Duquesne in [14] under a different forest model. The work of Duquesne keeps track of more of the genealogy of the random trees and forests, and that is not of primary concern to us here.

2.1. Basic Definitions. We define a forest \mathfrak{f} as a locally finite graph on some (possibly infinite) vertex set $V \subset \mathbb{N}$ which has a finite number of connected components which are themselves rooted planar trees. We note that the vertex set V is a subset of the natural number. Therefore, the vertices have some ordering inherited from the natural numbers. To distinguish this ordering from

the ordering on the natural numbers we write \prec instead of $<$, i.e for $v, w \in \mathfrak{f}$, we say $v \prec w$ if, as natural numbers, $v < w$. A priori, this has no special significance for the forests.

Each connected component is equipped with the graph distance. In this context, locally finite means that each vertex has finitely many adjacent vertices. In each of the connected components (which is a tree), say $\mathcal{T}_1, \dots, \mathcal{T}_M$, of \mathfrak{f} there will be a distinguished vertex $\rho_j \in \mathcal{T}_j$ called the root of the tree.

The height of a vertex $v \in \mathcal{T}_j$ is defined as the distance to the root ρ_j , and the height of v is denoted by either $\mathbf{ht}(v; \mathfrak{f})$ or $\mathbf{ht}(v; \mathcal{T}_j)$ which, depending on the context, should be clear. We also assume that the labeling of the vertices by elements of \mathbb{N} obeys the following ordering property, which is possible by the local finiteness condition:

(O1) If $\mathbf{ht}(v; \mathfrak{f}) < \mathbf{ht}(w; \mathfrak{f})$ then $v \prec w$.

For each vertex $v \in \mathcal{T}_j \setminus \{\rho_j\}$, there is a unique adjacent vertex w such that v and ρ_j are in separate connected components of $\mathcal{T}_j \setminus \{\rho_j\}$. We call this vertex w the parent of v , and is denoted by $\pi(v) = w$. The vertex v is called the child of w .

Next there will be some coloring of the vertices of \mathfrak{f} , which is just some map $\mathbf{cl} : \mathfrak{f} \rightarrow \{0, 1, \dots, N\}$ subject to the two conditions:

(C1) If $\mathbf{cl}(v) = 0$ then $\mathbf{cl}(\pi(v)) = 0$ or v is a root.

(C2) For each $j \in \{1, \dots, N\}$ the connected components of $\mathbf{cl}^{-1}(j) = \{v \in \mathfrak{f} : \mathbf{cl}(v) = j\}$ are finite.

(C3) If v and w are such that $0 < \mathbf{cl}(v) < \mathbf{cl}(w)$ and $\pi(v) = \pi(w)$ then $v \prec w$. If $\mathbf{cl}(w) = 0$ and $\mathbf{cl}(v) \neq 0$ with $\pi(v) = \pi(w)$ then $v \prec w$ as well.

Forests described above equipped with a coloring function satisfying (C1-C3) will be called a *colored forest*. We briefly describe what these conditions mean. Condition (C1) tells us that the vertices of color 0, have parents of color 0. In the context of [14] these are analogous to mutant vertices. Condition (C2) states that the subtrees of \mathfrak{f} where all vertices are of type $j \neq 0$ must be finite. This will be vital when describing the height processes. Condition (C3) just guarantees the children of a vertex have some relationship to the ordering of all vertices on the tree. Moreover, conditions (C3) implies that there is some breadth-first manner to which the vertices are labeled.

For each vertex v we define $\chi^j(v)$ as the number of type j children of vertex v .

Now consider a colored forest $(\mathfrak{f}, \mathbf{cl})$. Consider the connected components of $\mathbf{cl}^{-1}(j)$ which will be written as $\mathfrak{t}_0^j, \mathfrak{t}_1^j, \dots$. These connected components are trees, and are finite by assumption (C2). Moreover, for each component \mathfrak{t}_ℓ^j there is some unique vertex v_ℓ^j of minimal height. We say that these vertices v_ℓ^j 's are the j -roots of \mathfrak{t}_ℓ^j 's (or of \mathfrak{f}). We assume that we have indexed these vertices v_ℓ^j in a breadth-first manner so that if $\ell < m$ then $v_\ell^j \prec v_m^j$. This can be done since the forest \mathfrak{f} is locally finite, and so there are only a finite number of vertices in \mathfrak{f} of height at most h for any $h \geq 0$.

Lastly, since \mathfrak{t}_ℓ^j are finite we can describe a depth-first ordering of the tree in the obvious way. For $\#\mathfrak{t}_\ell^j = n$ the depth-first ordering of \mathfrak{t}_ℓ^j is w_1, \dots, w_n where $w_1 = v_\ell^j$ and given w_1, \dots, w_m the vertex w_{m+1} is

The first child of w_m if any; else
the next unexamined child of $\pi(w_m)$ if any; else
the next unexamined child of $\pi(\pi(w_m))$ if any; else,
and so on.

With this we can describe the j^{th} depth-first ordering of the forest \mathfrak{f} , which orders all type j vertices of \mathfrak{f} . Given two type j vertices v, w we say $v \prec^{(j)} w$ if

$v, w \in \mathfrak{t}_\ell^j$ and v appears before w in the depth-first ordering of \mathfrak{t}_ℓ^j ; or
 $v \in \mathfrak{t}_\ell^j$ and $w \in \mathfrak{t}_m^j$ with $\ell < m$.

Since the trees are locally finite and each tree \mathfrak{t}_ℓ^j has finite cardinality, we can enumerate all type j vertices in \mathfrak{f} by w_0^j, w_1^j, \dots where $w_\ell^j \prec^{(j)} w_m^j$ iff $\ell < m$. This means that the order of these trees will be applied in a breadth-first manner as well.

2.2. Processes on Forests. Given a finite tree \mathfrak{t} , with root ρ and depth-first ordering $(w_j; 0 \leq j < \#\mathfrak{t})$, we define the height process of \mathfrak{t} as

$$H_{\mathfrak{t}}(n) = \text{dist}(w_n, \rho).$$

Clearly, $H_{\mathfrak{t}}$ uniquely characterizes the tree \mathfrak{t} .

We also encode the information of the tree \mathfrak{t} in another way, called the Lukaciewicz path of \mathfrak{t} . If let $\chi(w)$ denote the number of children of a vertex w . The Lukaciewicz path of a tree \mathfrak{t} is denoted $(D_{\mathfrak{t}}(k); k = 0, \dots, \#\mathfrak{t})$ and is defined as

$$D_{\mathfrak{t}}(0) = 0, \quad D_{\mathfrak{t}}(k+1) = D_{\mathfrak{t}}(k) + \chi(w_k) - 1.$$

It is easily seen that $D_{\mathfrak{t}}(k) > -1$ for all $k = 0, \dots, \#\mathfrak{t} - 1$ and $D_{\mathfrak{t}}(\#\mathfrak{t}) = -1$. We recall from [27] without proof that the height process of a tree can be recovered from the Lukaciewicz path by

$$(9) \quad H_{\mathfrak{t}}(k) = \# \left\{ 0 \leq \ell < k : D_{\mathfrak{t}}(\ell) = \inf_{\ell \leq i \leq k} D_{\mathfrak{t}}(i) \right\}.$$

Now given a colored forest \mathfrak{f} , we fix a $j \in \{1, 2, \dots, N\}$. We let $(\mathfrak{t}_\ell^j; \ell \geq 1)$ be the connected components of $\mathbf{cl}^{-1}(j)$ constructed in the previous subsection. The roots of the trees \mathfrak{t}_ℓ^j are denoted by v_ℓ^j and the depth-first- j ordering of all vertices in $\mathbf{cl}^{-1}(j)$ is given by w_0^j, w_1^j, \dots . We define the j^{th} height process of the forest \mathfrak{f} as the process $H_{\mathfrak{f}}^j = (H_{\mathfrak{f}}^j(i); i \geq 0)$ where

$$(10) \quad H_{\mathfrak{f}}^j(i) = \text{dist}(w_i^j, v_\ell^j), \quad \text{when } w_i^j \in \mathfrak{t}_\ell^j.$$

We also define the j^{th} Lukaciewicz path of \mathfrak{f} as the process $D_{\mathfrak{f}}^j = (D_{\mathfrak{f}}^j(i); i \geq 0)$ where

$$D_{\mathfrak{f}}^j(0) = 0 \quad D_{\mathfrak{f}}^j(k+1) = D_{\mathfrak{f}}^j(k) + \chi^j(w_k^j) - 1,$$

where, as we recall, $\chi^j(w)$ is the number of type- j children of the vertex w . If $n_p^j = \#\mathfrak{t}_0^j + \dots + \#\mathfrak{t}_p^j$, then it is easy to see that for $i < \#\mathfrak{t}_{p+1}^j$ then

$$H_{\mathfrak{f}}^j(n_p^j + i) = H_{\mathfrak{t}_{p+1}^j}^j(i), \quad D_{\mathfrak{f}}^j(n_p^j + i) = D_{\mathfrak{t}_{p+1}^j}^j(i) - (p+1).$$

From here it follows that (9) remains true with $H_{\mathfrak{t}}$ (resp. $D_{\mathfrak{t}}$) replaced by $H_{\mathfrak{f}}^j$ (resp. $D_{\mathfrak{f}}^j$). We call the vector-valued processes $H_{\mathfrak{f}} = (H_{\mathfrak{f}}^1, \dots, H_{\mathfrak{f}}^N)$ the height process of \mathfrak{f} and $D_{\mathfrak{f}} := (D_{\mathfrak{f}}^1, \dots, D_{\mathfrak{f}}^N)$ the Lukaciewicz path of \mathfrak{f} .

We next define the height profile of a colored forest \mathfrak{f} . This is the \mathbb{Z}^N -valued process $Z_{\mathfrak{f}} = (Z_{\mathfrak{f}}^1, \dots, Z_{\mathfrak{f}}^N)$ defined by

$$(11) \quad Z_{\mathfrak{f}}^j(h) = \{v \in \mathfrak{f} : \mathbf{cl}(v) = j, \text{ and } \mathbf{ht}(v; \mathfrak{f}) = h\}.$$

We can further describe the $(i \rightarrow j)$ -height profile for $i \in \{0, 1, \dots, N\}$ and $j \in \{1, \dots, N\}$ as the process $Z_{\mathfrak{f}}^{i \rightarrow j} := (Z_{\mathfrak{f}}^{i \rightarrow j}(h); h \geq 0)$ defined by

$$(12) \quad Z_{\mathfrak{f}}^{i \rightarrow j}(h) = \{v \in \mathfrak{f} : \mathbf{cl}(v) = j, \mathbf{cl}(\pi(v)) = i, \mathbf{ht}(v; \mathfrak{f}) = h\},$$

with the understanding that if v is a root of \mathfrak{f} , then $\mathbf{cl}(\pi(v)) = 0$. Thus, we can see for each $j \in \{1, \dots, N\}$ and $h \geq 0$

$$Z_{\mathfrak{f}}^j(h) = \sum_{i=0}^N Z_{\mathfrak{f}}^{i \rightarrow j}(h).$$

We now define the *cumulative height profiles* as the processes $C_{\mathfrak{f}}^{i \rightarrow j} = (C_{\mathfrak{f}}^{i \rightarrow j}(h); h \geq 0)$ and $C_{\mathfrak{f}}^j = (C_{\mathfrak{f}}^j(h); h \geq 0)$ by

$$(13) \quad C_{\mathfrak{f}}^{i \rightarrow j}(h) = \sum_{m=0}^h Z_{\mathfrak{f}}^{i \rightarrow j}(m), \quad C_{\mathfrak{f}}^j(h) = \sum_{i=0}^N C_{\mathfrak{f}}^{i \rightarrow j}(h) = \sum_{m=0}^h Z_{\mathfrak{f}}^j(m).$$

We also describe a process which counts the number of type j -vertices whose parent is not of type j which have a height of at most h . We denote this by $I_{\mathfrak{f}}^j = (I_{\mathfrak{f}}^j(h); h \geq 0)$. We can see that as

$$I_{\mathfrak{f}}^j(h) = \sum_{i \neq j} C_{\mathfrak{f}}^{i \rightarrow j}(h).$$

It is easy to see that if \mathfrak{t}_{ℓ}^j are the connected components of $\mathbf{cl}^{-1}(j)$ with roots v_{ℓ}^j , then $\{v_{\ell}^j; \ell = 0, 1, \dots, I_{\mathfrak{f}}^j(h) - 1\}$ are all the j -roots of height at most h , because of the convention $Z_{\mathfrak{f}}^j(0) = Z_{\mathfrak{f}}^{i \rightarrow j}(0)$.

Altering some of the descriptions and definitions in [14, Pages 111-112] we define the j^{th} *left height process* $\overleftarrow{H}_{\mathfrak{f}}^j = (\overleftarrow{H}_{\mathfrak{f}}^j(i); i \geq 0)$ by

$$\overleftarrow{H}_{\mathfrak{f}}^j(i) = \mathbf{ht}(w_i^j; \mathfrak{f}).$$

We can see that if $w_i^j \in \mathfrak{t}_{\ell}^j$, then on the path from w_i^j to the root in the forest \mathfrak{f} lies the vertex v_{ℓ}^j , and hence we get

$$\overleftarrow{H}_{\mathfrak{f}}^j(i) = H_{\mathfrak{f}}^j(i) + \mathbf{ht}(v_{\ell}^j; \mathfrak{f}).$$

The height of v_{ℓ}^j is obtained by finding

$$\mathbf{ht}(v_{\ell}^j; \mathfrak{f}) = \inf\{h \geq 0 : I_{\mathfrak{f}}^j(h) > \ell\}.$$

This follows easily from a simple counting argument and the observation that we started indexing at 0 and we made the choice to say $Z_{\mathfrak{f}}^{0 \rightarrow j}(0) = Z_{\mathfrak{f}}^j(0)$.

Next, the ℓ for which $w_i^j \in \mathfrak{t}_{\ell}^j$ can be found from the Lukaciewicz path via

$$\ell = -D_{\mathfrak{f}}^j(i) := \inf\{D_{\mathfrak{f}}^j(m); m \leq i\}.$$

Hence

$$(14) \quad \overleftarrow{H}_{\mathfrak{f}}^j(i) = H_{\mathfrak{f}}^j(i) + \inf\{h \geq 0 : I_{\mathfrak{f}}^j(h) > -D_{\mathfrak{f}}^j(i)\}.$$

The last thing we describe is the breadth-first children functions for the forest \mathfrak{f} . We first, let $(\bar{w}_{\ell}^i; \ell \geq 0)$ denote the breadth-first labeling of all vertices of type i . We define the *breadth-first $i \rightarrow j$ children function* by

$$(15) \quad X_{\mathfrak{f}}^{i,j}(\ell) = \sum_{m=0}^{\ell-1} \chi^j(\bar{w}_m^i) - 1_{[i=j]} \quad i, j \neq 0$$

and

$$(16) \quad Y_{\mathfrak{f}}^j(\ell) = \sum_{m=0}^{\ell-1} \chi^j(\bar{w}_m^0), \quad i = 0.$$

We now observe, from [7], that $Z_{\mathfrak{f}}$ is the discrete solution to

$$(17) \quad Z_{\mathfrak{f}}^j(h+1) = Z_{\mathfrak{f}}^j(0) + \sum_{i=1}^N X_{\mathfrak{f}}^{i,j} \circ C_{\mathfrak{f}}^i(h) + Y_{\mathfrak{f}}^j(h), \quad C_{\mathfrak{f}}^j(h) = \sum_{\ell=0}^h Z_{\mathfrak{f}}^j(\ell).$$

2.3. Randomizing the Model. We now introduce some randomization into the model described above. We first fix $N(N+1)$ probability measures on $\mathbb{N}_0 = \{0, 1, \dots\}$, which, a priori, have no assumptions. These measures are labeled $\mu^{i,j}$ and ν^j for $i, j \in \{1, \dots, N\}$. We describe how to construct a colored forest \mathfrak{f} , by describing the roots and then growing the tree height-by-height. We won't focus on the labeling of vertices by elements of \mathbb{N} .

We start by fixing a vector $\vec{k} = (1, k_1, \dots, k_n)$. This will describe the roots of our forest. For ease of notation, we let $\mathfrak{f}_{-1} = \emptyset$.

- (1) There will be k_j roots of color j at height 0, and 1 root of color 0 at height 0, labeled subject to (O1). Call this \mathfrak{f}_0
- (2) For each $h \geq 0$, and $w \in \mathfrak{f}_h \setminus \mathfrak{f}_{h-1}$ of color i generate independent random variables $\chi^j(w)$ with distribution $\mu^{i,j}$ (when $i \neq 0$) or ν^j (when $i = 0$). If $i = 0$ then generate 1 vertex of type 0 as well.
- (3) For all $j \in [N]$ and $w \in \mathfrak{f}_h \setminus \mathfrak{f}_{h-1}$. At height $h+1$ add $\chi^j(w)$ children of type j and parent w , labeled subject to (C3). Call the resulting forest \mathfrak{f}_{h+1} .
- (4) Continue this process ad infinitum.

The resulting random colored forest will be defined as $\mathfrak{f} = \cup_{h \geq 0} \mathfrak{f}_h$.

We call the resulting forest a multi-type Galton-Watson immigration forest with offspring distributions $\boldsymbol{\mu} = (\mu^{i,j}; i, j \in [N])$ and immigration $\boldsymbol{\nu} = (\nu^j; j \in [N])$ started from \vec{k} individuals, which is abbreviated $\text{GWI}_{\vec{k}}(\boldsymbol{\mu}, \boldsymbol{\nu})$. For more information on multi-type Galton Watson processes see [35, 13, 7] and references therein. We can easily see that the height profile $Z_{\mathfrak{f}}$ of \mathfrak{f} is a multi-type Galton-Watson process with immigration. Indeed, if $(\xi_{\ell}^{i,j,h}; h \geq 0, \ell \geq 1)$ are i.i.d. with common distribution $\mu^{i,j}$ and $(\eta_h^j; h \geq 0)$ are i.i.d. with common distribution ν , then conditionally on $Z_{\mathfrak{f}}(h) = (z_1, \dots, z_N)$ we have

$$Z_{\mathfrak{f}}^j(h+1) = \eta_h^j + \sum_{i=1}^N \sum_{\ell=1}^{z_i} \xi_{\ell}^{i,j,h}.$$

We make the following crucial observation, for any $\text{GWI}_{\vec{k}}(\boldsymbol{\mu}, \boldsymbol{\nu})$ forest then

$$(18) \quad \left(D_{\mathfrak{f}}^j(m); m \geq 0 \right) \stackrel{d}{=} \left(X_{\mathfrak{f}}^{j,j}(m); m \geq 0 \right).$$

Indeed, this follows from the observation that both $(\chi^j(\bar{w}_m^j); m \geq 0)$ and $(\chi^j(w_m^j); m \geq 0)$ are both sequences of independent random variables with common distribution $\mu^{j,j}$.

As can be seen from equations (15) and (16), the processes $X_{\mathfrak{f}}^{i,j}$ and $Y_{\mathfrak{f}}^j$ are random walks.

3. OVERVIEW OF BRANCHING PROCESSES AND HEIGHT PROCESSES

Continuous state branching processes with or without immigration are an object of much study. Continuous state branching (CB for short) processes are Feller processes on $[0, \infty]$ with cemetery states of 0 and ∞ . An immigration component can be added to obtain a continuous state branching process with immigration (CBI for short) which is a Feller process on $[0, \infty]$ with only ∞ as the only absorbing state, excluding the situation where the immigration rate is 0. Kawazu and Watanabe in [21] show that CBI processes are uniquely determined by their Laplace transforms, i.e. if Z is a CBI process then there exists functions ψ and ϕ , for all $\lambda > 0$

$$(19) \quad -\log \mathbb{E}_x [\exp\{-\lambda Z_t\}] = xu(t, \lambda) + \int_0^t \phi(u(s, \lambda)) ds$$

where u is the unique solution to the integral equation

$$u(t, \lambda) + \int_0^t \psi(u(s, \lambda)) ds = \lambda.$$

The function ψ is called the *branching mechanism* and the function ϕ is called the *immigration rate*. We say that Z is a CBI(ψ, ϕ) process of short, and if we wish to specify the starting position we will write $\text{CBI}_x(\psi, \phi)$. Kawazu and Watanabe further classified what the function ϕ must satisfy. Prior

to their work, Silverstein [33] classified the form of the functions ψ which can appear. Summarizing, these the functions must satisfy

$$(20) \quad \begin{aligned} \psi(\lambda) &= -\kappa + \alpha\lambda + \beta\lambda^2 + \int_{(0,\infty)} (e^{-\lambda r} - 1 + \lambda r 1_{[r < 1]}) \pi(dr) \\ \phi(\lambda) &= \kappa' + \alpha'\lambda - \int_{(0,\infty)} (e^{-\lambda r} - 1) \bar{\pi}(dr) \end{aligned},$$

where $\kappa, \beta, \kappa', \alpha' \geq 0$, $\alpha \in \mathbb{R}$, π and $\bar{\pi}$ are Radon measures on $(0, \infty)$ with $\int_{(0,\infty)} (1 \wedge r^2) \pi(dr) < \infty$

and $\int_{(0,\infty)} (1 \wedge r) \bar{\pi}(dr) < \infty$.

From (20) there are two Lévy processes X and Y which are characterized by ψ and ϕ . That is, there exists a spectrally positive (i.e. no negative jumps) Lévy process X and a subordinator Y such that for each $\lambda > 0$

$$(21) \quad \mathbb{E}[\exp\{-\lambda X_t\}] = \exp\{t\psi(\lambda)\}, \quad \mathbb{E}[\exp\{-\lambda Y_t\}] = \exp\{-t\phi(\lambda)\}.$$

For more information of Lévy processes, see, for example, Bertoin's monograph [3].

There does exist a path-wise relationship between CBI processes and these Lévy processes X and Y , which is due to Caballero, Pérez Garmendia and Uribe Bravo in [6]. Their work generalized the result accredited to Lamperti [25], but first proved by Silverstein in [33]. The work in [6] shows given independent X and Y , there exists a unique càdlàg solution to the integral equation

$$(22) \quad Z_t = x + X_{\int_0^t Z_s ds} + Y_t,$$

and Z is a $\text{CBI}_x(\psi, \phi)$ processes. When Y is zero the process X is stopped at $-x$ and the map is invertible.

3.1. Multi-type Branching Processes. There are numerous generalizations of CBI processes, including allowing immigration from outside sources. We call these new processes multi-type continuous state branching processes (resp. with immigration), written as MCB (resp. MCBI). These were described in a two-dimensional system in [35]. A more general picture of multi-type branching processes was described in [13] as a particular example of so-called affine processes.

We'll shortly state the characterization of multi-type continuous state branching process with immigration found in [13, Theorem 2.7]. An \mathbb{R}_+^N -valued process Z is a multi-type continuous state branching process with immigration if, for each $\lambda \in \mathbb{R}_+^N$ and $x \in \mathbb{R}_+^N$

$$(23) \quad \mathbb{E}_x[\exp\{-\langle \lambda, Z_t \rangle\}] = \exp\left\{-\langle x, u(t, \lambda) \rangle - \int_0^t \tilde{\phi}(u(s, \lambda)) ds\right\},$$

where $u = (u_1, \dots, u_N)$ is a solution

$$(24) \quad u_i(t, \lambda) + \int_0^t \tilde{\psi}_i(u(s, \lambda)) ds = \lambda_i$$

for a collection of functions $\tilde{\psi}_i$. The functions $\tilde{\phi}$ and $\tilde{\psi}_i$, $i \in [N]$, are also characterized by

$$(25) \quad \begin{aligned} \tilde{\psi}_i(\lambda) &= \beta_i \lambda_i^2 - \langle A \vec{e}_i, \lambda \rangle + \int_{\mathbb{R}_+^N \setminus \{0\}} (e^{-\langle \lambda, z \rangle} - 1 + \lambda_i (1 \wedge z_i)) \pi^i(dz) \\ \tilde{\phi}(\lambda) &= \langle \delta, \lambda \rangle - \int_{\mathbb{R}_+^N \setminus \{0\}} (e^{-\langle \lambda, z \rangle} - 1) \pi_0(dz) \end{aligned},$$

where \vec{e}_i is the i^{th} standard basis element in \mathbb{R}^N . In the above equations, $(\beta, \delta, A, \pi^0, \boldsymbol{\pi})$ are *admissible parameters*. Namely, they satisfy

- (i) $\beta = (\beta_1, \dots, \beta_N)$ and $\delta = (\delta_1, \dots, \delta_N)$ where $\beta_i, \delta_i \geq 0$.
- (ii) $A = (\alpha_{i,j})_{i,j \in [N]}$ is an $N \times N$ matrix with $\alpha_{j,j} \leq 0$ and $\alpha_{i,j} \geq 0$ where $i \neq j$.

(iii) $\pi = (\pi_1, \dots, \pi_N)$ is a collection of N Radon measures on $\mathbb{R}_+^N \setminus \{0\}$ such that

$$\int_{\mathbb{R}_+^N \setminus \{0\}} \left[\|z\| \wedge \|z\|^2 + \sum_{i \neq j} z_i \right] \pi_j(dz) < \infty \quad \text{for all } j \in [N].$$

(iv) π_0 is a Radon measure on $\mathbb{R}_+^N \setminus \{0\}$ such that $\int_{\mathbb{R}_+^N \setminus \{0\}} (1 \wedge \|z\|) \pi_0(dz) < \infty$.

As can be observed from above, the work of [13] gives the bijection between MCBI processes Z and a collection of $(N+1)$ Lévy processes X^i, Y for $i \in [N]$ and each take values in \mathbb{R}^N . That is, the functions $\tilde{\psi}_i$ and $\tilde{\phi}$ are related to X^i and Y by

$$(26) \quad \mathbb{E} [\exp \{-\langle \lambda, X_1^i \rangle\}] = e^{\tilde{\psi}_i(\lambda)} \quad \mathbb{E} [\exp \{-\langle \lambda, Y_1 \rangle\}] = e^{-\tilde{\phi}(\lambda)}.$$

However, the authors of [13] do not give a path-wise relationship. There is a path-wise representation, due to Caballero, Pérez Garmendia and Uribe Bravo in [7], which extended the result in a Ph.D. thesis of Gabrielli [17] which had some additional technical assumptions. In the former work, the authors show that given these $(N+1)$ Lévy processes, there exists a solution Z to the following initial value problem

$$(27) \quad Z_t^j = x_j + \sum_{i=1}^N X^{i,j}(C_t^i) + Y_t^j, \quad C_t^j = \int_0^t Z_s^j ds,$$

where $X^i = (X^{i,1}, X^{i,2}, \dots, X^{i,N})$ and $Y = (Y^1, Y^2, \dots, Y^N)$. As the notation may suggest, the process $Z_t = (Z_t^1, \dots, Z_t^N)$ is a multi-type continuous state branching process associated with the $(N+1)$ Lévy processes. The work of [7] shows that for any MCBI process, we can find a decomposition of the form (27).

3.2. Height Processes. Throughout the remainder of the work, we will be working under stricter assumptions on the Laplace transforms of the Lévy processes we consider. We will state those assumptions in this section along with what they entail. As they pertain to height processes, we will state several facts about Lévy processes without proof or explicit reference. The results can be found in Bertoin's monograph [3], especially Chapter VII.

The continuous time height process $H = (H_t; t \geq 0)$ is the continuous time analog of equation (9) where the random walk D_\dagger is replaced with a spectrally positive Lévy process $X = (X_t; t \geq 0)$. We do restrict the type of Lévy processes we consider in order to guarantee certain desirable properties, for example continuity, of the height processes.

We assume that ψ satisfies equation (2), where ψ_j is replaced with ψ (along with corresponding replacements for α, β, π). The integral condition guarantees the almost sure extinction of a CBI $(\psi, 0)$ process (see [19]) and implies either $\beta > 0$ or $\int_0^1 r \pi(dr) = \infty$ and, hence, that X has paths of infinite variation almost surely.

The analog of (9) is the ψ -height process H , which is associated with X , is defined to give a meaningful measure to the set

$$\{s \in [0, t] : X_{s-} = \inf_{s \leq r \leq t} X_r\}.$$

This is made precise by a time-reversal argument in [27], which we will not go over in full detail. For a relatively brief overview of the height process H , we recommend Section 3.1 in [14] and for a more detailed overview see Chapter 1 in [15].

We do state, however, that when $\beta > 0$ the height process is

$$H_t = \frac{1}{\beta} \text{Leb} \left\{ \inf_{s \leq r \leq t} X_r; s \leq t \right\}.$$

Moreover, under the conditions on ψ in (2), the height process H is continuous, see [15, Theorem 1.4.3].

The Ray-Knight theorem of [15] is now briefly recalled. The local time $L = (L_t^a; a, t \geq 0)$ of H , is defined by the approximation formula ([15, Proposition 1.3.3])

$$\lim_{\varepsilon \downarrow 0} \mathbb{E} \left[\sup_{s \in [0, t]} \left| \frac{1}{\varepsilon} \int_0^s 1_{[a < H_s \leq a + \varepsilon]} - L_t^a \right| \right].$$

For $\tau_r = \inf\{t > 0 : X_s = -r\}$, the Ray-Knight theorem states that for $Z_a = L_{\tau_r}^a$ the process Z is a CBI($\psi, 0$) process started from r . The work was extended by Duquesne [14] and Lambert [24] to include some immigration mechanism.

4. THE WEAK SOLUTION

In this section we describe how we construct solutions to the stochastic equation in (5), under the assumptions of Theorem 1.1 We will prove the theorem, by appealing to several lemmas which follow from results in the existing literature.

4.1. Preliminary Lemmas. We now describe the assumptions we will make on the GWI forests we analyze. We suppose that $(\mu_p; p \geq 1)$ and $(\nu_p; p \geq 1)$ are a sequence of probability measures and $(\gamma_p; p \geq 1)$ are any increasing sequence of non-negative integers. For each $i, j \in [N]$ and $p \geq 1$ we let $(\xi_\ell^{i,j,p}; \ell \geq 0)$ (resp. $(\eta_\ell^{j,p}; \ell \geq 0)$) denote i.i.d. sequences with common distribution $\mu_p^{i,j}$ (resp. ν_p^j). We let $g^{j,p}$ denote the generating function of $\mu_p^{j,j}$ and iteratively define $g_n^{j,p} = g_{n-1}^{j,p} \circ g^{j,p}$ where $g_0^{j,p} = id$. We make the following assumptions

(A1) Jointly for all $i, j \in [N]$ we have the following convergence in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$

$$\begin{aligned} \left(\frac{1}{p} \sum_{\ell=0}^{\lfloor p\gamma_p t \rfloor - 1} (\xi_\ell^{i,j,p} - 1_{[i=j]}) ; t \geq 0 \right) &\Longrightarrow (\mathbf{X}_t^{i,j} ; t \geq 0) \\ \left(\frac{1}{p} \sum_{\ell=0}^{\lfloor \gamma_p t \rfloor - 1} \eta_\ell^{j,p} ; t \geq 0 \right) &\Longrightarrow (\mathbf{Y}_t^j ; t \geq 0) \end{aligned}$$

(A2) The processes $\mathbf{X}_t^{i,j}$ and \mathbf{Y}_t^j above are independent Lévy processes where $\mathbf{Y}_t^j = \delta_j t$ and for $i \neq j$, $\mathbf{X}_t^{i,j} = \alpha_{i,j} t$ for $\delta_j > 0$ and $\alpha_{i,j} \geq 0$. The Laplace exponents ψ_j of $\mathbf{X}^{j,j}$ satisfy (2).

(A3) The generating functions satisfy

$$\liminf_{p \rightarrow \infty} g_{\lfloor \delta \gamma_p \rfloor}^{j,p}(0) > 0 \quad \forall \delta > 0.$$

If each assumption above is satisfied, we say that μ_p and ν_p satisfy assumption (A).

We remark that if we are given Laplace exponents $(\psi_j; j \in [N])$ which are admissible then we can construct a sequence of probability measures μ_p and ν_p which satisfy assumption (A). Indeed, we can take $\xi_\ell^{i,j,p}$ for $i \neq j$ and $\eta_\ell^{j,p}$ to be independent Poisson random variables with appropriate means.

Throughout the sequel we will write the subscript p for processes on forests as opposed to \mathfrak{f}_p . For example, we will write $Z_p^j(h)$ as opposed to $Z_{\mathfrak{f}_p}^j(h)$. With this new notation, we can state the following lemma. A rigorous proof is omitted since it follows from equation (17) along with Theorem 2 and Theorem 3 in [7]. We also recall that $\underline{f}(t) = \inf_{s \leq t} f(s)$.

Lemma 4.1. *Suppose μ_p and ν_p satisfy assumption (A). Suppose that $\vec{k}_p = (1, k_{1,p}, \dots, k_{N,p})$ where $k_{j,p}/p \rightarrow x_j \geq 0$ as $p \rightarrow \infty$. Let $(\mathfrak{f}_p; p \geq 1)$ denote a sequence of $\text{GWI}_{\vec{k}_p}(\mu_p, \nu_p)$ forests. Then, in the Skorohod J_1 topology on $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^N)$ the following convergence holds*

$$(28) \quad \left(\frac{1}{p} Z_p([\gamma_p v]) ; v \geq 0 \right) \Longrightarrow (\mathbf{Z}_v ; v \geq 0),$$

where $\mathbf{Z} = (\mathbf{Z}^1, \dots, \mathbf{Z}^N)$ is the unique solution to

$$(29) \quad \mathbf{Z}_v^j = x_j + \sum_{i=1}^N \mathbf{X}_{\mathbf{C}_v^i}^{i,j} + \mathbf{Y}_v^j, \quad \mathbf{C}_v^j = \int_0^v \mathbf{Z}_s^j ds.$$

Moreover, the convergence is joint with the convergence in assumption **(A1)**

We now observe that the height process for the forest, along with the Lukaciewicz path converge jointly as well. We state this as the following lemma, which is a simple application of Corollary 2.5.1 in [15] and using the fact [15, Equation (1.7)] holds in our situation by **(A3)**.

Lemma 4.2. *Suppose the conditions on μ_p, ν_p and \vec{k}_p hold. Then, in the $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^3)$ the following joint convergence holds*

$$(30) \quad \left(\left(\frac{1}{p} D_p^j([p\gamma_p t]), \frac{1}{\gamma_p} H_p^j([p\gamma_p t]), \frac{1}{p} D_p^j([p\gamma_p t]) \right); t \geq 0 \right) \Longrightarrow \left(\left(\tilde{\mathbf{X}}_t^{j,j}, \mathbf{H}_t^j, \inf_{s \leq t} \tilde{\mathbf{X}}_s^{j,j} \right); t \geq 0 \right)$$

where $\tilde{\mathbf{X}}^{j,j} \stackrel{d}{=} \mathbf{X}^{j,j}$ and \mathbf{H}_t^j is a ψ_j -height process constructed from $\tilde{\mathbf{X}}^{j,j}$.

Remark 4.1. We note that by [15, Lemma 1.3.2] that $-\inf_{s \leq t} \tilde{\mathbf{X}}_s^{j,j} = \ell_t^j$ where ℓ_t^j is the local time at level 0 and time t of the process \mathbf{H}^j defined as the L^1 -limit in (4).

4.2. Convergence of the left-height process. We start with a useful convergence lemma, along with an observation on the path-wise behavior of the limiting process.

Lemma 4.3. *Suppose μ_p, ν_p satisfy assumption **(A)** and the $\vec{k}_p = (1, k_{1,p}, \dots, k_{N,p})$ satisfies $k_{j,p}/p \rightarrow x_j \geq 0$ as $p \rightarrow \infty$. Then the following convergence holds, jointly with the convergences in equations (28) and assumption **(A1)**,*

$$(31) \quad \left(\frac{1}{p} I_p^j([\gamma_p v]); v \geq 0 \right) \Longrightarrow (\mathbf{U}_v^j; v \geq 0),$$

where

$$\mathbf{U}_v^j = x_j + \sum_{i \neq j} \alpha_{i,j} \mathbf{C}_v^i + \delta_j v, \quad \mathbf{C}_v^j = \int_0^v \mathbf{Z}_s^j ds.$$

Moreover, the process \mathbf{U}^j is strictly increasing and $\mathbf{U}_t^j \rightarrow \infty$ as $t \rightarrow \infty$.

Proof. We begin by noting that

$$(32) \quad C_p^{i \rightarrow j}(h) = \begin{cases} X_p^{i,j}(C_p^i(h-1)) & : i \neq 0, j \\ k_{j,p} + Y_p^j(h) & : i = 0 \end{cases}.$$

Indeed, all type i vertices of height at most $h-1$ are enumerated by w_ℓ^i for $\ell = 0, 1, \dots, C_p^i(h-1) - 1$. We get the above discrete time change observation by noting that every vertex of type j and height at most h as a parent of height at most $h-1$.

We now observe the following

$$\begin{aligned} \frac{1}{p\gamma_p} C_p^j([\gamma_p v]) &= \frac{1}{p\gamma_p} \sum_{h=0}^{[\gamma_p v]} Z_p^j(h) = \int_0^{[\gamma_p v]} \frac{1}{p\gamma_p} Z_p^j([s]) ds \\ &= \int_0^{[\gamma_p v]/\gamma_p} \frac{1}{p} Z_p^j([\gamma_p s]) ds. \end{aligned}$$

Hence, the we get $\left(\frac{1}{p\gamma_p} C_p^j([\gamma_p v]); v \geq 0 \right) \Longrightarrow (\mathbf{C}_v^j; v \geq 0)$ in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$, where \mathbf{C}_v^j is as in (29). This convergence, as can easily be seen, holds jointly with the convergences of (28) and **(A1)**.

The desired convergence in (31) thus follows from equation (32) and standard time-change results. The statement about \mathbf{U}^j being strictly increasing follows from the observations that $\mathbf{Z}^i \geq 0$ a.s. and $\delta_j > 0$. \square

We have now gathered all of the pieces necessary for proving the existence of a solution to equation (5). Instead of stating the proof of Theorem 1.1, we prove the theorem below, which is easily seen to imply both Theorem 1.1 and Corollary 1.2.

Theorem 4.4. *Suppose μ_p, ν_p satisfy assumption (A) and \vec{k}_p satisfy $k_{j,p}/p \rightarrow x_j$ as $p \rightarrow \infty$. Then following convergence holds jointly*

$$(33) \quad \left(\left(\frac{1}{\gamma_p} \overleftarrow{H}_p^j([p\gamma_p t]); j \in [N], t \geq 0 \right); \left(\frac{1}{p} Z_p^j([\gamma_p v]); j \in [N], v \geq 0 \right) \right) \\ \implies \left(\left(\overleftarrow{\mathbf{H}}_t^j; j \in [N], t \geq 0 \right); \left(\mathbf{L}_\infty^v(\overleftarrow{\mathbf{H}}^j); j \in [N], v \geq 0 \right) \right).$$

where

$$\overleftarrow{\mathbf{H}}_t^j = \mathbf{H}_t^j + \inf \left\{ x > 0 : \mathbf{U}_x^j > \ell_t^j \right\}.$$

Remark 4.2. We observe that Theorem 1.1, and hence Corollary 1.2 follows from the above theorem. Indeed, Theorem 1.1 follows from the observation that $U_0^j = x_j$ and $\frac{d}{dx} \mathbf{U}_x^j = \delta + \sum_{i \neq j} \lambda_{i,j} \mathbf{L}_\infty^x(\mathbf{H}^i)$ and hence the process

$$\mathbf{J}_t^j := \inf \left\{ x > 0 : \mathbf{U}_x^j > \ell_t^j \right\},$$

must satisfy

$$\mathbf{J}_t^j = \int_{\tau_{x_j}^j}^{t \vee \tau_{x_j}^j} \frac{1}{\delta_j + \sum_{i \neq j} \alpha_{i,j} \mathbf{L}_\infty^j(\mathbf{H}^i)} d\ell_s^j.$$

Proof of Theorem 4.4. The proof follows quite easily from the various lemmas we have proved and discrete processes which we have defined. We observe from (14) the following holds

$$\begin{aligned} \frac{1}{\gamma_p} \overleftarrow{H}_p^j([p\gamma_p t]) &= \frac{1}{\gamma_p} H_p^j([p\gamma_p t]) + \frac{1}{\gamma_p} \inf \left\{ h : I_p^j(h) > -\underline{D}_p^j([p\gamma_p t]) \right\} \\ &= \inf \left\{ \frac{h}{\gamma_p} : \frac{1}{p} I_p^j(h) > \frac{-1}{p} \underline{D}_p^j([p\gamma_p t]) \right\} \\ &= \frac{1}{\gamma_p} H_p^j([p\gamma_p t]) + \inf \left\{ x : \frac{1}{p} I_p^j([\gamma_p x]) > -\frac{1}{p} \underline{D}_p^j([p^2 t]) \right\}. \end{aligned}$$

We now recall a result of Whitt [36]. It states that on the space $\mathbb{D}^\uparrow(\mathbb{R}_+, \mathbb{R}_+) := \{g \in \mathbb{D}(\mathbb{R}_+, \mathbb{R}_+) : \sup g = \infty\}$, the first passage time is continuous at each strictly increasing function. In terms of functions, the function

$$F : \mathbb{D}^\uparrow(\mathbb{R}_+, \mathbb{R}_+) \rightarrow \mathbb{D}(\mathbb{R}_+, \mathbb{R}_+) \quad \text{by} \quad F(f)(t) = \inf \{s > 0 : f(s) > t\}$$

is continuous at each f which is strictly increasing and diverge to infinity.

Hence, by Lemma 4.3 we know that \mathbf{U}_x^j is almost surely continuous, diverging towards infinity and is strictly increasing and hence we can conclude in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}_+)$ that

$$\left(\inf \left\{ x : \frac{1}{p} I_p^j([\gamma_p x]) > v \right\}; v \geq 0 \right) \implies \left(\inf \{x : \mathbf{U}_x^j > v\}; v \geq 0 \right).$$

Since $-\underline{D}_p$ is increasing, Lemma 4.2 and standard weak convergence arguments imply

$$\left(\inf \left\{ x : \frac{1}{p} I_p^j([\gamma_p x]) > -\frac{1}{p} \underline{D}_p^j([p^2 t]) \right\}; t \geq 0 \right) \implies \inf \left\{ x > 0 : \mathbf{U}_x^j > \ell_t^j \right\}.$$

By a tightness argument, we can take this convergence to be joint with the convergence of $\frac{1}{\gamma_p} H_p^j([p\gamma_p t])$ towards \mathbf{H}_t^j . Hence we prove the convergence of the left-height process.

The rescaled process $\frac{1}{p} Z_p^j([\gamma_p v])$ converges by Lemma 4.1. The joint convergence follows from similar tightness arguments as the proof of Corollary 2.5.1 in [15]. See also Theorem 1.5 in [14] for a similar result which relies on the proof of Corollary 2.5.1. This completes the proof of the result. \square

4.3. Consequences of Theorem 4.4. In this subsection we discuss the consequences of Theorem 4.4. We observe that by Lemma 4.1 above and Theorem 1 in [7] the local time of the processes $\widehat{\mathbf{H}}^j$ is a multi-type continuous state branching process. We state this as the following proposition:

Proposition 4.5. *The processes $\mathbf{Z}_v^j = \mathbf{L}_\infty^v(\widehat{\mathbf{H}}^j)$, $j \in [N]$ form a multi-type continuous state branching process with immigration determined by equation (29).*

Stochastic equations for \mathbf{Z} are possible thanks to the results of [2]. We now use that work and focus on the Brownian situation of Corollaries 1.2 and 1.3. In particular we assume that $\mathbf{X}^{j,j}$ have Laplace exponents of the form

$$\psi_j(\lambda) = -\alpha_{j,j}\lambda + \beta_j\lambda^2$$

where $\beta_j > 0$ and $\alpha_{j,j} \leq 0$. In turn, the process $\mathbf{X}_t^{j,j} = \sqrt{2\beta_j}B_t^j + \alpha_{j,j}t$ is a Brownian motion with negative drift. The computations in Section 5 of [2] imply that

$$(34) \quad \mathbf{Z}_v^j = x_j + \int_0^v \left(\delta_j + \sum_{i=1}^N \alpha_{i,j} \mathbf{Z}_s^i \right) ds + \int_0^v \sqrt{2\beta_j \mathbf{Z}_s^j} dW_v^j,$$

for an \mathbb{R}^N -valued Brownian motion (W^1, \dots, W^N) . From here Corollary 1.3 follows.

5. GENERALIZED SQUARED BESSEL PROCESS

We begin with defining the set $\mathbb{D}^{\uparrow\uparrow} \subset \mathbb{D}(\mathbb{R}_+, \mathbb{R})$ as the set of all unbounded strictly increasing càdlàg functions F with $F(0) = 0$. We equip $\mathbb{D}^{\uparrow\uparrow}$ with the usual Skorokhod J_1 topology, which coincides with the subspace topology.

We now make the simple observations about solutions to (8).

Proposition 5.1. *For any $F \in \mathbb{D}^{\uparrow\uparrow}$, there is weak existence and weak uniqueness of solutions to the stochastic equation (8). The solution \mathbf{Z} is càdlàg and the set of discontinuities of \mathbf{Z} are contained in the discontinuities of F , i.e.*

$$\{v : \mathbf{Z}_v \neq \mathbf{Z}_{v-}\} \subset \{v : F(v) \neq F(v-)\}.$$

Proof. The existence and uniqueness implicit in the following proposition follows from an application of Corollary 1 in [6]. The discontinuity observation follows from the fact that $\int_0^v \sqrt{\mathbf{Z}_s} dW_s$ is almost surely continuous. \square

We now state a definition analogous to Definition 2 in [16].

Definition 2. We call the solution \mathbf{Z} a *generalized squared Bessel process with integrated dimension F* started from x . We call such a solution \mathbf{Z} a GBESQ $_x^F$ -process, and denote its law on the Skorokhod space $\mathbb{D} := \mathbb{D}(\mathbb{R}_+, \mathbb{R}_+)$ by \mathbb{Q}_x^F .

There are well known additivity properties of squared Bessel processes and for generalized squared Bessel process under some smoothness assumptions on F . We can get rid of many of those smoothness assumptions with the following proposition

Proposition 5.2. *Let $F, G \in \mathbb{D}^{\uparrow\uparrow}$ and $x, y \geq 0$. Then*

$$\mathbb{Q}_x^F * \mathbb{Q}_y^G = \mathbb{Q}_{x+y}^{F+G},$$

where $*$ denotes the convolution of measures. Moreover, if $\{F_n; n \in \mathbb{N}\} \subset \mathbb{D}^{\uparrow\uparrow}$ and $x_n \geq 0$ are such that $F_n \rightarrow F \in \mathbb{D}^{\uparrow\uparrow}$ in the J_1 topology and $x_n \rightarrow x \geq 0$ then

$$\mathbb{Q}_{x_n}^{F_n} \Longrightarrow \mathbb{Q}_x^F$$

weakly as probability measures on \mathbb{D} .

Remark 5.1. It is well known that the Skorokhod space \mathbb{D} , equipped with the usual J_1 topology, is not a topological group under addition. See, for example, [5, Problem 12.2]. As Whitt remarks in [36, Section 4], addition is not continuous in *any* of topologies Skorokhod defines in his original paper [34]. Although addition is not continuous from $\mathbb{D} \times \mathbb{D} \rightarrow \mathbb{D}$, it is a measurable map and so convolution is still a well-defined operation.

Proof. The proof of the convolution result follows along the same lines as the proof of Theorem XI.1.2 in [32]. The only difference is taking careful note that addition is not a continuous function from $\mathbb{D}^2 \rightarrow \mathbb{D}$, and so we must use the fact that this map is measurable. This can be found in Theorem 4.1 of [36].

To prove the continuity claim we observe that there exists a Brownian motion B such that a GBESQ $_x^F$ process \mathbf{Z} satisfies

$$\mathbf{Z}_v = x + 2B_{\int_0^v \mathbf{Z}_s ds} + F(v).$$

But

$$\int_0^\infty \frac{1}{B_s \vee 0} ds = \infty \quad \text{almost surely.}$$

The continuity claim thus follows from applications of Proposition 3 and Theorem 3 in [6]. \square

5.1. Existence of a Height Process. In this section we will establish Theorem 1.4. We use a forest model and prove this by selecting an appropriate offspring distribution μ in the finite case and then take limits. We do not prove a more universal result, which would require additional technical assumptions on the offspring distribution analogous to those found in (3).

Therefore, we fix a probability measure μ on \mathbb{N}_0 such that if $(\xi_j; j = 0, 1, \dots)$ are an i.i.d. sequence of random variables with common law μ such that

$$\left(\frac{1}{p} \sum_{\ell=0}^{\lfloor p^2 t \rfloor - 1} (\xi_j - 1); t \geq 0 \right) \implies (2B_t; t \geq 0),$$

where B_t is a standard Brownian motion. We also define $F_p(v) = \left\lceil pF\left(\frac{v}{p}\right) \right\rceil$ and observe that

$$\lim_{n \rightarrow \infty} \frac{1}{p} F_p(\lfloor pv \rfloor) = F(v)$$

in the Skorokhod J_1 topology. Indeed this follows from the assumption that F is increasing, Theorem 3.1 in [36] and the observation that $\frac{1}{p} F_p(\lfloor pv \rfloor)$ is the composition $\Phi_p \circ F \circ \Phi_p$ where $\Phi_p(x) = \frac{\lfloor px \rfloor}{p}$ which converges locally uniformly to $\Phi(x) = x$.

We now define the forest \mathfrak{f}_p to be constructed in a breadth-first manner as in Section 2.3 where there is only the 0 and 1 types of individuals. We start with $\lfloor px \rfloor$ roots of type 1, and at generation h the type 0 individuals begets (the deterministic number) $F_n(h+1) - F_n(h)$ of type 1 individuals. Since we only analyze the type 1 individuals in the forest \mathfrak{f}_p , we omit the superscripts when describing processes on the forests and use the subscript p instead of \mathfrak{f}_p . For example, we write $Z_p(h) = Z_{\mathfrak{f}_p}^1(h)$. We also observe that

$$I_p(h) = \lfloor px \rfloor + Y_p(h) = \lfloor px \rfloor + F_p(h).$$

We now move onto proving Theorem 1.4.

Proof of Theorem 1.4. We begin with the observation that under the above assumptions on μ that we have the following weak convergence in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^3)$ due to [15, Corollary 2.5.1]:

$$\left(\left(\frac{1}{p} D_p(\lfloor p^2 t \rfloor), \frac{1}{p} H_p(\lfloor p^2 t \rfloor), \frac{1}{p} \underline{D}_p(\lfloor p^2 t \rfloor) \right); t \geq 0 \right) \implies \left((2B_t, \beta_t, -L_t^0(\beta)); t \geq 0 \right)$$

where B is a Brownian motion, β is the reflected Brownian motion $\beta_t = B_t - \underline{B}_t$ and $L_t^0(\beta)$ is the local time at time t and level 0 of β . Moreover, since $I_p(h) = F_p(h)$, we have the convergence

$\frac{1}{p}I_p([pt]) \longrightarrow x + F(t)$. Hence, the left-height process of the forest \mathfrak{f}_p satisfies the following limit

$$\left(\frac{1}{p}\overleftarrow{H}_p([p^2t]); t \geq 0\right) \Longrightarrow \left(\overleftarrow{\mathbf{H}}_t; t \geq 0\right),$$

where

$$\overleftarrow{H}_t = \beta_t + F^{-1}(\ell_{(t-x)_+})$$

with $F^{-1}(x) = \inf\{u : F(u) > x\}$ is the right-continuous inverse of F .

Moreover, by (17) we have

$$Z_p(h+1) = [px] + X_p \circ C_p(h) + Y_p(h)$$

where $X_p \stackrel{d}{=} D_p$ and $Y_p(h) = F_p(h)$. Thus, by the work in [6], and the assumptions on μ we have the following weak convergence

$$\left(\frac{1}{p}Z_p([pv]); v \geq 0\right) \Longrightarrow (\mathbf{Z}_v; v \geq 0)$$

where \mathbf{Z} is the unique càdlàg solution to

$$\mathbf{Z}_v = x + 2\tilde{B}_{\int_0^v \mathbf{Z}_s ds} + F(v)$$

where \tilde{B} is a standard Brownian motion.

Moreover, a tightness argument similar to [15, Corollary 2.5.1] gives the joint convergence

$$(35) \quad \left(\left(\frac{1}{p}\overleftarrow{H}_p([p^2t]); t \geq 0\right), \left(\frac{1}{p}Z_p([pv]); v \geq 0\right) t \geq 0\right) \Longrightarrow \left(\left(\overleftarrow{\mathbf{H}}_t\right), \left(\mathbf{L}_\infty^v(\overleftarrow{\mathbf{H}}); v \geq 0\right)\right).$$

As previously argued, $\mathbf{L}_\infty^v(\overleftarrow{\mathbf{H}})$ is equal in distribution to the unique càdlàg solution

$$\begin{aligned} \mathbf{Z}_v &= x + 2\tilde{B}_{\int_0^v \mathbf{Z}_s ds} + F(v) \\ &= x + 2 \int_0^t \sqrt{\mathbf{Z}_s} dW_s + F(v) \end{aligned}$$

as desired. \square

A similar argument to the proofs of Theorems 1.4 and 1.1 can easily yield the following theorem. The proof of Theorem 1.1 remains quite similar, with the exception that the immigrants are added according to a *deterministic* immigration rate. The full proof is omitted, but is analogous to the above.

Theorem 5.3. *Let $\beta^1, \beta^2, \dots, \beta^N$ denote N independent reflected Brownian motions with drift $\alpha_{j,j}$ for some $\alpha_{j,j} \leq 0$. For each $j \in [N]$, let ℓ_t^j denote the local time at zero of β^j . Suppose, for each $j \in [N]$, that g_j is a non-negative function such that $G_j(t) := \int_0^t g_j(s) ds$ is strictly increasing and diverging towards infinity and $x_j \geq 0$. Also, for $i \neq j$ let $\alpha_{i,j} \geq 0$. Then the following holds:*

(1) *There exist a weak solution $(\overleftarrow{\mathbf{H}}_t^j; j \in [N], t \geq 0)$ to the system of stochastic equations*

$$\begin{cases} \mathbf{H}_t^j = \beta_t^j + \mathbf{J}_t^j \\ \mathbf{J}_t^j = \int_{\tau_{x_j}}^{t \vee \tau_{x_j}} \left\{ g_j(\mathbf{J}_s^j) + \sum_{i \neq j} \alpha_{i,j} \mathbf{L}_\infty^{\mathbf{J}_s^j}(\overleftarrow{\mathbf{H}}^i) \right\}^{-1} d\ell_s^j \end{cases}.$$

(2) *Define $\mathbf{Z}_v^j = \mathbf{L}_\infty^v(\overleftarrow{\mathbf{H}}^j)$. Then $(\mathbf{Z}_v^j; j \in [N])$ is a weak solution to the system of stochastic differential equations:*

$$d\mathbf{Z}_v^j = 2\sqrt{\mathbf{Z}_v^j} dW_v^j + \left(g_j(v) + \sum_{i=1}^N \alpha_{i,j} \mathbf{Z}_v^i \right) dv, \quad \mathbf{Z}_0^j = x_j,$$

for independent Brownian motions W^j .

6. A GORIN-SHKOLNIKOV TYPE RESULT

In a recent paper [18] Gorin and Shkolnikov obtained, as a corollary of a result about random matrices, that

$$\int_0^1 e_t dt - \frac{1}{2} \int_0^\infty (L_1^v(e))^2 dv \stackrel{d}{=} \mathcal{N}\left(0, \frac{1}{12}\right),$$

where $e = (e_t; t \in [0, 1])$ is a standard Brownian excursion and $(L_1^v(e); v \geq 0)$ is its total local time of e at level v . The result was proved using path-wise properties of the local time of an excursion by Hariya [20] and still further expanded to reflected Brownian bridges in [23].

We now briefly recall the generalization of the identity in [9]. Suppose $H = (H_t; t \geq 0)$ is a continuous ψ -height process as in [15], i.e. ψ is of the form found in (20) with $\kappa = 0$, $\alpha, \beta \geq 0$ and π is a Radon measure with $\int_{(0, \infty)} (r \wedge r^2) \pi(dr) < \infty$. Further assume that ψ is conservative, i.e.

$$\int_1^\infty \frac{1}{\psi(\lambda)} < \infty.$$

Let ℓ_t denote the local time at level 0 of the process H . Fix $x \geq 0$ and $\delta > 0$, then set

$$\mathbf{H}_t = H_t + \frac{1}{\delta}(\ell_t - x)_+$$

and let $\mathbf{L}_\infty^v := \mathbf{L}_\infty^v(\mathbf{H})$ denote the local time of \mathbf{H} at level v . The process \mathbf{L}_∞^v was shown to be a continuous state branching process with branching mechanism ψ and immigration function $\phi(\lambda) = \delta\lambda$ by [24] and [14]. We finally define

$$\mathbf{V}_r = \inf \left\{ h \geq 0 : \int_0^h \mathbf{L}_\infty^v(\mathbf{H}) dv > r \right\}$$

and \mathbf{X} to be a Lévy process with Laplace exponent ψ as in (21). Corollary 3.3 in [9] becomes the equation

$$\left(\delta \int_0^\infty \mathbf{H}_t 1_{[\mathbf{H}_t < \mathbf{V}_r]} dr - \int_0^{\mathbf{V}_r} (\mathbf{L}_\infty^v)^2 dv; r \geq 0 \right) \stackrel{d}{=} \left(-xr - \int_0^r \mathbf{X}_s ds; r \geq 0 \right)$$

6.1. Extension to Systems of Equations. Now that we have a solution to the system of equations in (5), we can extend the Gorin-Shkolnikov identity to systems of equations. We first establish the notation used throughout this subsection. We fix some family of admissible $(\psi_j; j \in [N])$. We assume that $\overleftarrow{\mathbf{H}}^j$ and $\mathbf{L}_\infty^v(\overleftarrow{\mathbf{H}}^j)$ are as in Theorem 1.1 and Theorem 4.4, with the constants as described in those Theorems. We also let \mathbf{V}_r^j be defined as

$$(36) \quad \mathbf{V}_r^j = \inf \left\{ h \geq 0 : \int_0^h \mathbf{L}_\infty^v(\overleftarrow{\mathbf{H}}^j) dv > r \right\}.$$

Let $X = (X^1, \dots, X^N)$ denote an \mathbb{R}^N -valued Lévy process where X^j has Laplace exponent $(-\psi_j)$ for the collection of admissible ψ_j . Define the \mathbb{R}^N -valued random fields $\mathbf{G} = (\mathbf{G}^1, \dots, \mathbf{G}^N) = (\mathbf{G}(\vec{r}); \vec{r} \in \mathbb{R}_+^N)$ and $\mathbf{S} = (\mathbf{S}(\vec{r}); \vec{r} \in \mathbb{R}_+^N)$ by

$$(37) \quad \begin{aligned} \mathbf{G}^j(\vec{r}) &= \delta_j \int_0^\infty \overleftarrow{\mathbf{H}}_t^j 1_{[\overleftarrow{\mathbf{H}}_t^j < \mathbf{V}_{r_j}^j]} dt - \int_0^{\mathbf{V}_{r_j}^j} (\mathbf{L}_\infty^v(\overleftarrow{\mathbf{H}}^j))^2 dv + \sum_{i \neq j} \alpha_{i,j} \int_0^\infty \int_0^\infty 1_{[\overleftarrow{\mathbf{H}}_s^i < \overleftarrow{\mathbf{H}}_t^j < \mathbf{V}_{r_j}^j]} ds dt, \\ \mathbf{S}^j(\vec{r}) &= -x_j r_j - \int_0^{r_j} X_t^j dt \end{aligned}$$

where $\vec{r} = (r_1, \dots, r_N)$.

We can now state the extension.

Theorem 6.1. *As random fields,*

$$(\mathbf{G}(\vec{r}); \vec{r} \in \mathbb{R}_+^N) \stackrel{d}{=} (\mathbf{S}(\vec{r}); \vec{r} \in \mathbb{R}_+^N)$$

Proof. We let $\mathbf{Z}_v = (\mathbf{Z}_v^1, \dots, \mathbf{Z}_v^N)$ be a MCBI process as in Lemma 4.1. We observe that $\mathbf{Z}^j \stackrel{d}{=} \mathbf{L}_\infty^v(\overleftarrow{\mathbf{H}}^j)$. Since $\mathbf{Z}_v^j \stackrel{d}{=} \mathbf{L}_\infty^v(\mathbf{H}^j)$, we can and do abuse notation and write $\mathbf{V}_r^j = \inf\{h > 0 : \int_0^h \mathbf{Z}_s^j ds > r\}$.

We rearrange equation (29) to obtain

$$\mathbf{Z}_v^j - \delta_j v - \sum_{i \neq j} \alpha_{i,j} \int_0^v \mathbf{Z}_s^i ds = x_j + \mathbf{X}^{j,j}(\mathbf{C}_v^j)$$

It follows that, jointly for $j \in [N]$,

$$\begin{aligned} & \int_0^{\mathbf{V}_{r_j}^j} \left\{ (\mathbf{Z}_v^j)^2 - \delta_j v \mathbf{Z}_v^j - \sum_{i \neq j} \alpha_{i,j} \mathbf{Z}_v^i \int_0^v \mathbf{Z}_s^i ds \right\} dv \\ &= \int_0^{\mathbf{V}_{r_j}^j} [x_j + \mathbf{X}^{j,j}(\mathbf{C}_v^j)] \mathbf{Z}_v^j dv \\ &= \int_0^{r_j} [x_j + \mathbf{X}^{j,j}t] dt \stackrel{d}{=} -\mathbf{S}^j(\vec{r}). \end{aligned}$$

We now observe that since the local times of $\overleftarrow{\mathbf{H}}^j$ and the MCBI process \mathbf{Z} are equal in law that we have the following equality in distribution as processes

$$\begin{aligned} & \int_0^{\mathbf{V}_{r_j}^j} \left\{ (\mathbf{Z}_v^j)^2 - \delta_j v \mathbf{Z}_v^j - \sum_{i \neq j} \alpha_{i,j} \mathbf{Z}_v^i \int_0^v \mathbf{Z}_s^i ds \right\} dv \\ & \stackrel{d}{=} \int_0^{\mathbf{V}_{r_j}^j} (\mathbf{L}_\infty^v(\overleftarrow{\mathbf{H}}^j))^2 dv - \delta_j \int_0^{\mathbf{V}_{r_j}^j} v \mathbf{L}_\infty^v(\overleftarrow{\mathbf{H}}^j) dv - \sum_{i \neq j} \alpha_{i,j} \int_0^{\mathbf{V}_{r_j}^j} \mathbf{L}_\infty^v(\overleftarrow{\mathbf{H}}^j) \int_0^v \mathbf{L}_\infty^u(\overleftarrow{\mathbf{H}}^i) du dv. \end{aligned}$$

This gives the desired claim as

$$\begin{aligned} & \int_0^h \int_0^v \mathbf{L}_\infty^v(\mathbf{H}^j) \mathbf{L}_\infty^u(\mathbf{H}^i) du dv \\ &= \int_0^\infty dt \left\{ 1_{[\overleftarrow{\mathbf{H}}_t^j < h]} \int_0^{\mathbf{H}_t^j} \mathbf{L}_\infty^u(\overleftarrow{\mathbf{H}}^i) du \right\} \\ &= \int_0^\infty dt \left\{ 1_{[\overleftarrow{\mathbf{H}}_t^j < h]} \int_0^\infty 1_{[\overleftarrow{\mathbf{H}}_s^i < \overleftarrow{\mathbf{H}}_t^j]} ds \right\} \\ &= \int_0^\infty \int_0^\infty 1_{[\overleftarrow{\mathbf{H}}_s^i < \overleftarrow{\mathbf{H}}_t^j < h]} ds dt. \end{aligned}$$

□

6.2. Gorin-Shkolnikov of General Perturbed Brownian Motion. Let us consider the semimartingale $\overleftarrow{\mathbf{H}}$ described in Theorem 1.4, for some function $F \in \mathbb{D}^{\uparrow\uparrow}$. Denote its local time by $\mathbf{L}_\infty^v(\overleftarrow{\mathbf{H}})$. We observe that in the case where $F(v) = \delta v$, the results of [9] imply

$$\delta \int_0^\infty \overleftarrow{\mathbf{H}}_t 1_{[\overleftarrow{\mathbf{H}}_t < \mathbf{V}_r]} dt - \int_0^{\mathbf{V}_r} (\mathbf{L}_\infty^v(\overleftarrow{\mathbf{H}}))^2 dv \stackrel{d}{=} -xr - 2 \int_0^r B_s ds.$$

More generally, we observe that \mathbf{L}_∞^v is equal in law to the unique càdlàg solution \mathbf{Z} to the integral equation

$$\mathbf{Z}_v = x + 2B \int_0^v \mathbf{Z}_s ds + F(v).$$

Rearranging and integrating, we have

$$\begin{aligned} \int_0^{\mathbf{V}_r} (\mathbf{L}_\infty^v(\overleftarrow{\mathbf{H}}) - F(v)) \mathbf{L}_\infty^v(\overleftarrow{\mathbf{H}}) &\stackrel{d}{=} \int_0^{\mathbf{V}_r} (\mathbf{Z}_v - F(v)) \mathbf{Z}_v dv \\ &= \int_0^{\mathbf{V}_r} (x + 2B_{\int_0^v \mathbf{z}_s ds}) \mathbf{Z}_v dv \\ &= \int_0^r (x + 2B_s) ds. \end{aligned}$$

Using the occupation time formula, we observe that this implies the following theorem:

Theorem 6.2. *Let $\beta = (\beta_t; t \geq 0)$ denote a reflected Brownian motion and let ℓ_t denote its local time at time t and level 0. Let F be any strictly increasing càdlàg function and $x \geq 0$. Define*

$$\overleftarrow{\mathbf{H}} = \beta_t + F^{-1}((\ell_t - x)_+).$$

Let $\mathbf{L}_\infty^v(\overleftarrow{\mathbf{H}})$ denote the local time at level v of $\overleftarrow{\mathbf{H}}$ and let $\mathbf{V}_r = \inf\{t > 0 : \int_0^t \mathbf{L}_\infty^v(\overleftarrow{\mathbf{H}}) dv > r\}$. Then

$$\int_0^\infty F(\overleftarrow{\mathbf{H}}_r) 1_{[\overleftarrow{\mathbf{H}}_r < \mathbf{V}_t]} dr - \int_0^{\mathbf{V}_t} (\mathbf{L}_\infty^v(\overleftarrow{\mathbf{H}}))^2 dv \stackrel{d}{=} -xt + 2 \int_0^t W_s ds$$

for a Brownian motion W .

7. ADDITIONAL COMMENTS

The forms of numerous theorems in this work suggest the following two conjectures. The first is stated as:

Conjecture 7.1. *Let $f : \mathbb{R}_+ \times C([0, \infty)) \rightarrow [\varepsilon, \infty)$ for some $\varepsilon > 0$ be a progressively measurable functional. Then there exists a weak solution to the stochastic equation*

$$\begin{cases} \mathbf{H}_t = \beta_t + \mathbf{J}_t \\ \mathbf{J}_t = \int_{\tau_x}^{t \vee \tau_x} \frac{1}{f((\mathbf{L}_\infty^\bullet(\mathbf{H})); \mathbf{J}_s)} d\ell_s, \end{cases} .$$

Moreover $\mathbf{Z}_v := \mathbf{L}_\infty^v(\mathbf{H})$ is a weak solution to the stochastic differential equation

$$d\mathbf{Z}_v = 2\sqrt{\mathbf{Z}_v} dW_v + f(\mathbf{Z}_\bullet, v) dv.$$

Another natural question to ask is whether or not the existence of equation (5) requires the admissible assumption on the Laplace exponents ψ_j . We state it as:

Conjecture 7.2. *For any ψ_j -height process where ψ_j satisfy (2), there exists a solution to (5).*

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UNIVERSITY OF WASHINGTON
Email address: djclancy@uw.edu