HIDDEN GEMS: ALDOUS'S *Brownian excursions, critical random graphs and the multiplicative coalescent*

David Clancy

Oct. 5, 2020

Erdős-Rényi random graphs

The Erdős-Rényi random graph $G(n, p)$ is the graph on the vertices

$$V = [n] := \{1, 2, \cdots, n\}$$

and the edge connecting i and j is *independently* added with probability p:

$$\mathbb{P}(\{i, j\} \text{ is an edge}) = p$$

Erdős-Rényi random graphs

$G(n, p)$ gives a simple model of a random graph.

Erdős-Rényi random graphs

$G(n,p)$ gives a simple model of a random graph.
Probabilistic method: $G(n,p)$ can be used to prove the existence of certain combinatorial objects without a direct construction.

Erdős-Rényi random graphs

$G(n,p)$ gives a simple model of a random graph.
Probabilistic method: $G(n,p)$ can be used to prove the existence of certain combinatorial objects without a direct construction.
Almost equivalent to the study of certain model in epidemiology. The edges represent disease transmission in the Reed-Frost model.

CRITICALITY OF $G(n, p)$

Basic properties of $G(n, p)$:

- Total number of edges is $\text{Bin}\left(\binom{n}{2}, p\right)$, $E[\#\text{total edges}] \approx \frac{n^2 p}{2}$

Criticality of $G(n, p)$

Basic properties of $G(n, p)$:

- Total number of edges is $\text{Bin}\left(\binom{n}{2}, p\right)$, $E[\#\text{total edges}] \approx \frac{n^2 p}{2}$
- Expect $(n-1)p$ other vertices to share an edge with each vertex j.

Criticality of $G(n, p)$

We expect each vertex to have $\approx np$ neighbors.

Criticality of $G(n, p)$

We expect each vertex to have $\approx np$ neighbors.
What happens when $p = p(n) = \frac{c}{n}$?

Criticality of $G(n, p)$

We expect each vertex to have $\approx np$ neighbors.
What happens when $p = p(n) = \frac{c}{n}$?

Theorem 1 (Erdős & Rényi [ER60]).

$G_n = G(n, c/n)$ for some constant $c \in (0, \infty)$. As $n \to \infty$:

- *$c > 1$: the largest component of G_n is of order n, the second largest component of G_n is of order $\log n$;*
- *$c < 1$: the largest component of G_n is of order $\log n$;*
- *$c = 1$: the largest two components of G_n are both order $n^{2/3}$.*

Criticality of $G(n, p)$

We expect each vertex to have $\approx np$ neighbors.
What happens when $p = p(n) = \frac{c}{n}$?

Theorem 1 (Erdős & Rényi [ER60]).

$G_n = G(n, c/n)$ for some constant $c \in (0, \infty)$. As $n \to \infty$:

- *$c > 1$: the largest component of G_n is of order n, the second largest component of G_n is of order $\log n$;*
- *$c < 1$: the largest component of G_n is of order $\log n$;*
- *$c = 1$: the largest two components of G_n are both order $n^{2/3}$.*

Something interesting happens at $p = 1/n$.

Critical window

What happens near $p(n) = \frac{1}{n}$?

Critical window

What happens near $p(n) = \frac{1}{n}$?
More formally: What happens when

$$p(n) = \frac{1 \pm \varepsilon(n)}{n} \qquad \text{where } \varepsilon(n) \to 0 \text{ as } n \to \infty?$$

Critical window

What happens near $p(n) = \frac{1}{n}$?
More formally: What happens when

$$p(n) = \frac{1 \pm \varepsilon(n)}{n} \qquad \text{where } \varepsilon(n) \to 0 \text{ as } n \to \infty?$$

Too many results to summarize for general ε_n windows.
We'll focus on is

$$p = \frac{1 + \lambda n^{-1/3}}{n} = n^{-1} + \lambda n^{-4/3}, \qquad \lambda \in \mathbb{R}.$$

WHY $n^{-1} + \lambda n^{-4/3}$?

Theorem 2 (Bollobás '84, Łuczak, Pittel, Wierman '94).

1. *Bollobás [Bol84]: If* $p = n^{-1} + n^{-(1+\gamma)}$ *for* $\gamma \in (0, 1/3)$ *then* $o(n^{2/3})$ *vertices appear in components that aren't trees or uni-cycles (graphs with 1 cycle).*

WHY $n^{-1} + \lambda n^{-4/3}$?

Theorem 2 (Bollobás '84, Łuczak, Pittel, Wierman '94).

1. *Bollobás [Bol84]: If* $p = n^{-1} + n^{-(1+\gamma)}$ *for* $\gamma \in (0, 1/3)$ *then* $o(n^{2/3})$ *vertices appear in components that aren't trees or uni-cycles (graphs with 1 cycle).*
2. *Ł-P-W [ŁPW94]: If* $p = n^{-1} + \lambda n^{-4/3}$ *then all components in* $G(n, p)$ *have at most* ξ_n *surplus edges added, and* ξ_n *is bounded in probability as* $n \to \infty$*.*

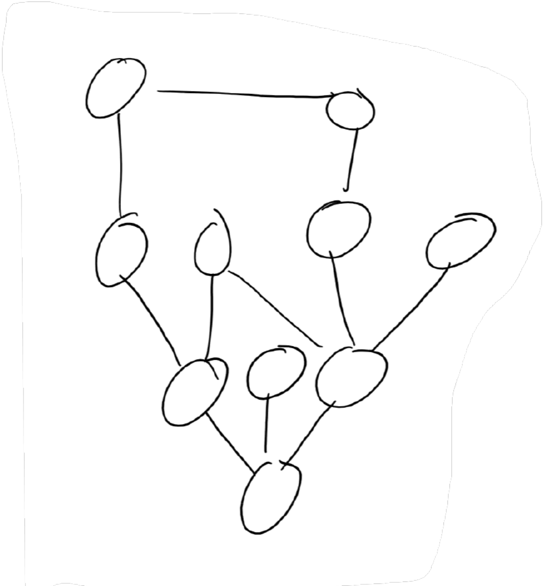
Surplus edges: $\#\text{edges} - \#\text{vertices} + 1$, the number of edges you have to remove from a graph in order to form a tree.

Breadth-first tree in a component

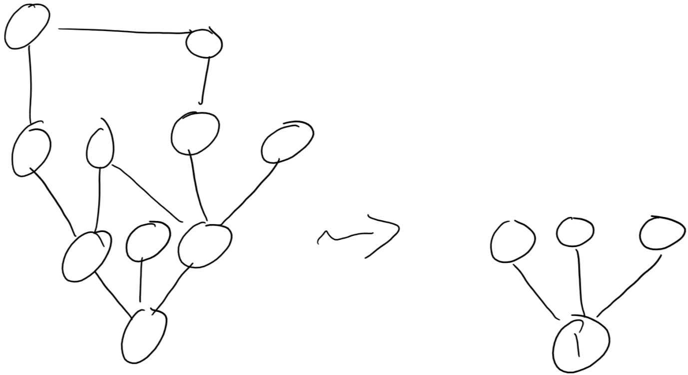
Aldous explores the graph $G(n, n^{-1} + \lambda n^{-4/3})$ via a *breadth-first walk*.

Breadth-first tree in a component

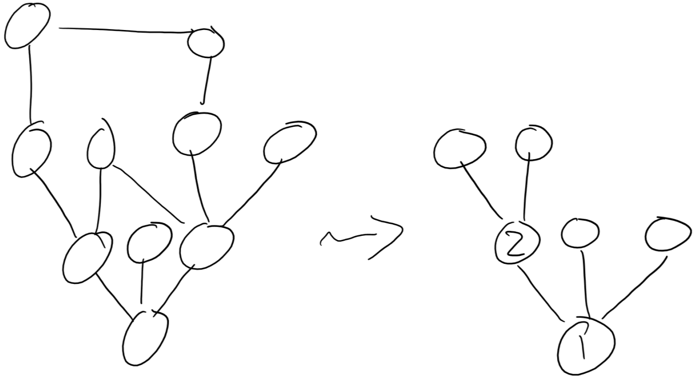
Aldous explores the graph $G(n, n^{-1} + \lambda n^{-4/3})$ via a *breadth-first walk*.



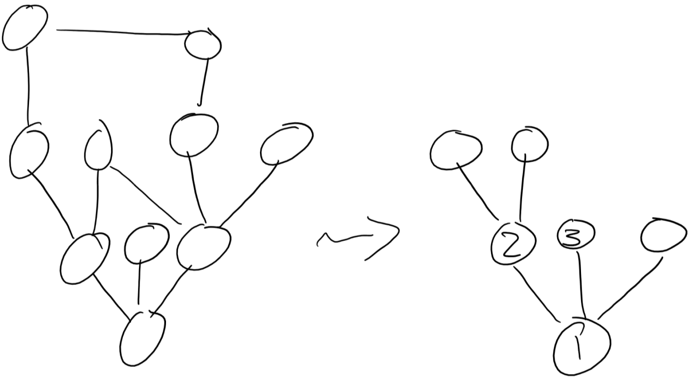
Breadth-first tree in a component



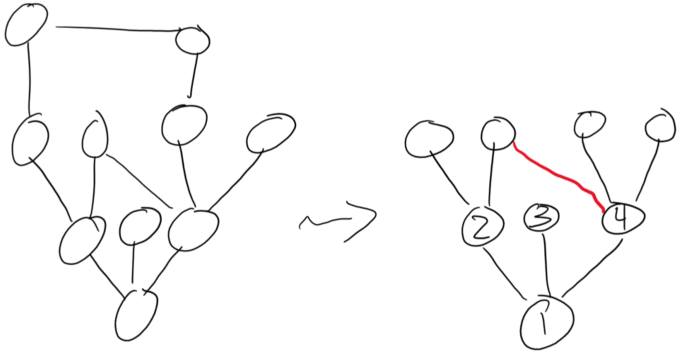
Breadth-first tree in a component



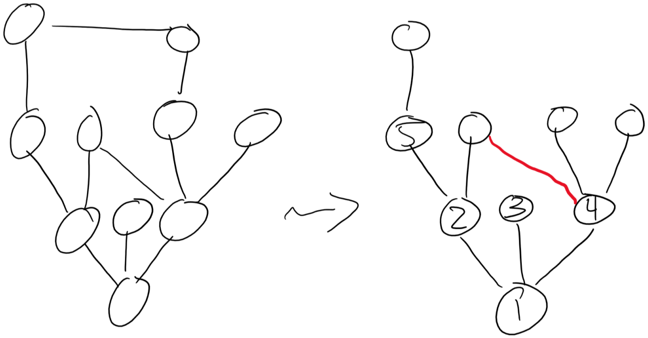
Breadth-first tree in a component



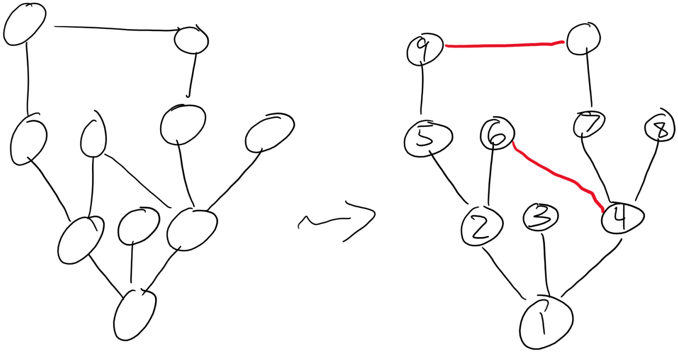
Breadth-first tree in a component



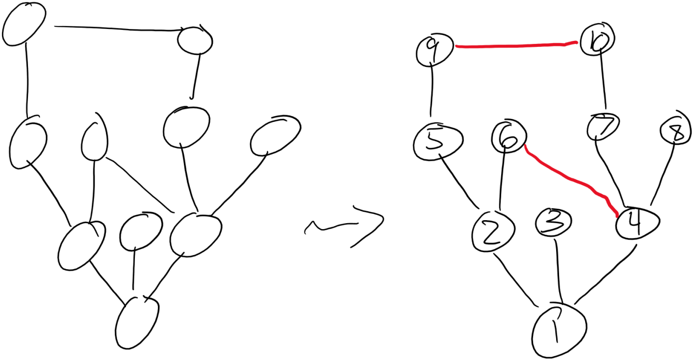
Breadth-first tree in a component



Breadth-first tree in a component



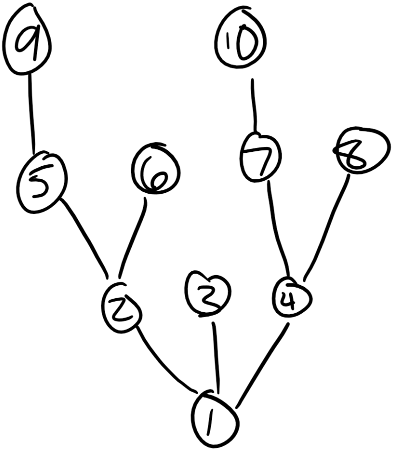
Breadth-first tree in a component



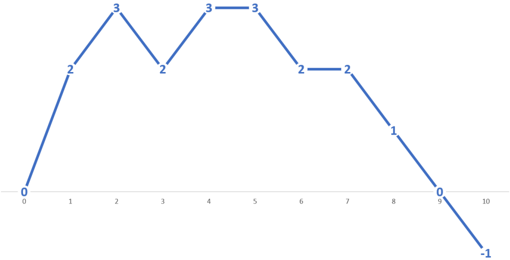
Aldous gets *breadth-first walk*, $(X_n(k); k = 0, 1, \cdots)$.

Aldous gets *breadth-first walk*, $(X_n(k); k = 0, 1, \cdots)$.
The increments satisfy:

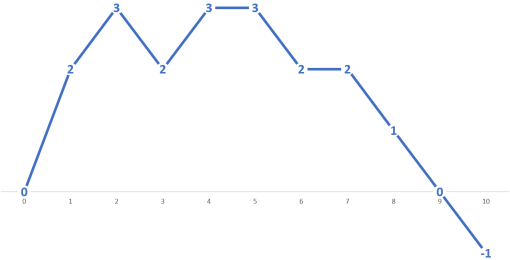
$$X_n(k) - X_n(k-1) = \#\,(\text{vertices discovered by vertex } k) - 1.$$



Vertex	Discovered Vertices
1	2, 3, 4
2	5, 6
3	None
4	7, 8
5	9
6	None
7	10
8	None
9	None
10	None

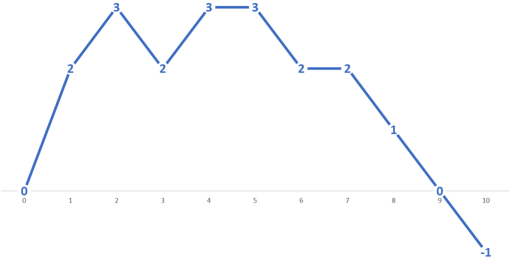


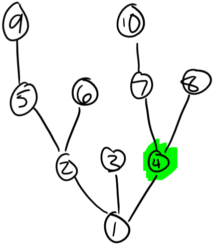
Vertex	Discovered Vertices
1	2, 3, 4
2	5, 6
3	None
4	7, 8
5	9
6	None
7	10
8	None
9	None
10	None

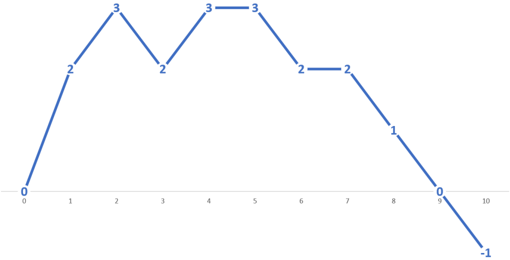


Vertex	Discovered Vertices
1	2, 3, 4
2	5, 6
3	None
4	7, 8
5	9
6	None
7	10
8	None
9	None
10	None

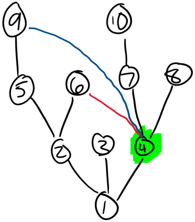
The excursion starts at zero, and ends at -1.


0
2
3
2
3
3
2
2
1
0
-1
0
1
2
3
4
5
6
7
8
9
10

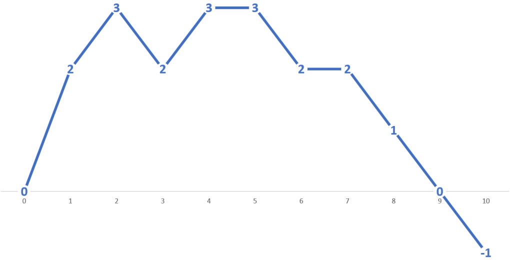

9
10
5
6
7
8
2
3
4
1

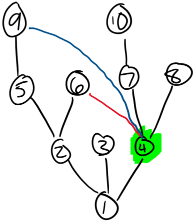


0
2
3
2
3
3
2
2
1
0
-1
0
1
2
3
4
5
6
7
8
9
10

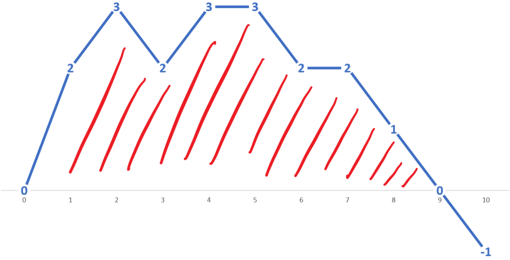


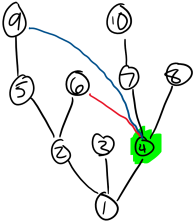
9
10
5
6
7
8
2
3
4
1





The edges $\{4, 6\}$ (in red) and $\{4, 5\}$ (not drawn) are allowable surplus edges, but $\{4, 9\}$ (in blue) is not.





The edges $\{4,6\}$ (in red) and $\{4,5\}$ (not drawn) are allowable surplus edges, but $\{4,9\}$ (in blue) is not.
The total number of allowable surplus edges is roughly the area under the excursion on the left.

PROPERTIES OF $X_n(k)$:

1. Component sizes are $T_n(j) - T_n(j-1)$ where $T_n(j) = \min\{k : X_n(k) = -j\}$

Properties of $X_n(k)$:

1. Component sizes are $T_n(j) - T_n(j-1)$ where $T_n(j) = \min\{k : X_n(k) = -j\}$
2. The number of surplus edges (red edges before) in a component is approximately

$$\approx \mathrm{Bin}\left(A_n(j), p\right)$$

$$A_n(j) = \text{area under the } j^{\text{th}} \text{ excursion}$$

Properties of $X_n(k)$:

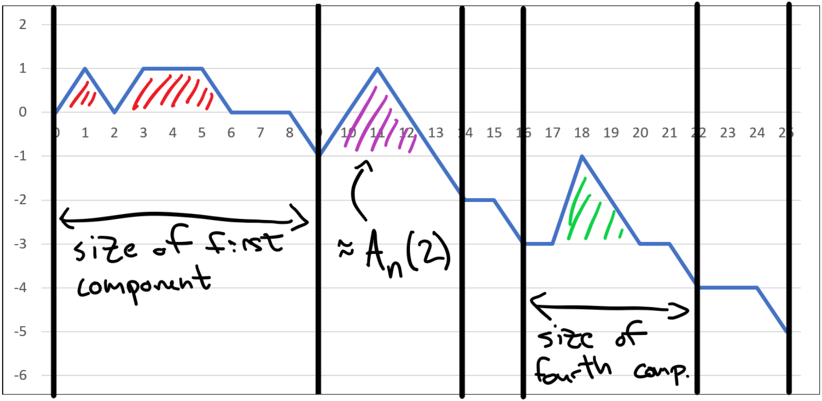
1. Component sizes are $T_n(j) - T_n(j-1)$ where $T_n(j) = \min\{k : X_n(k) = -j\}$
2. The number of surplus edges (red edges before) in a component is approximately

$$\approx \operatorname{Bin}(A_n(j), p)$$

$$A_n(j) = \text{area under the } j^{\text{th}} \text{ excursion}$$

Spencer [Spe97] has a wonderful (and short!) paper on why you should expect to see the area term $A_n(j)$.

Properties of $X_n(k)$:



Scaling Limits

Recall $p = 1/n$ then the two largest components are order $n^{2/3}$.

Scaling Limits

Recall $p = 1/n$ then the two largest components are order $n^{2/3}$.
Size of components are $T_n(j+1) - T_n(j)$ are order $n^{2/3}$.

Scaling Limits

Recall $p = 1/n$ then the two largest components are order $n^{2/3}$.
Size of components are $T_n(j+1) - T_n(j)$ are order $n^{2/3}$.

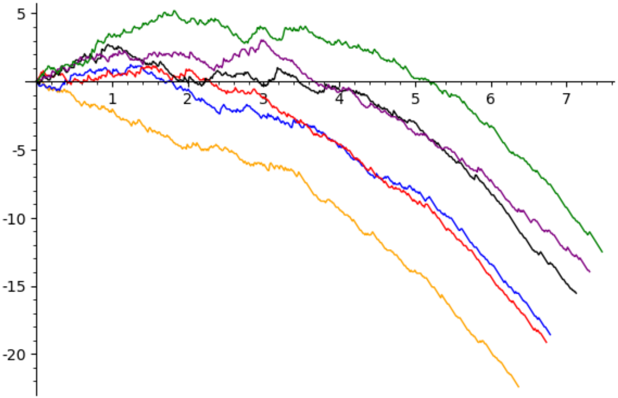


Figure: Brownian scaling: Simulations of $n^{-1/3}X_n(\lfloor n^{2/3}t \rfloor)$ for $\lambda \in \{0, 1, 2, 3, 4, 5\}$ and $n = 700$.

Scaling Limits

Recall $p = 1/n$ then the two largest components are order $n^{2/3}$.
Size of components are $T_n(j+1) - T_n(j)$ are order $n^{2/3}$.
Brownian scaling is $c^{-1/2}W(ct)$.

Theorem 3 (Aldous '97 [Ald97]).

As $n \to \infty$

$$n^{-1/3} X_n(\lfloor n^{2/3} t \rfloor) \xrightarrow{d} W(t) + \lambda t - \frac{1}{2} t^2$$

as processes, and W a standard Brownian motion.

Scaling Limits: Components

$C_n(1), C_n(2), \cdots$ components of $G(n,p)$ with

$$\#C_n(1) \geq \#C_n(2) \geq \cdots$$

Scaling Limits: Components

$C_n(1), C_n(2), \cdots$ components of $G(n,p)$ with

$$\#C_n(1) \geq \#C_n(2) \geq \cdots$$

Each of these components is encoded by an excursion like process: $(X_{n,i}(k); k \geq 0)$ encodes $C_n(i)$.

Scaling Limits: Components

$C_n(1), C_n(2), \cdots$ components of $G(n,p)$ with

$$\#C_n(1) \geq \#C_n(2) \geq \cdots$$

Each of these components is encoded by an excursion like process: $(X_{n,i}(k); k \geq 0)$ encodes $C_n(i)$.

Theorem 4 (It's complicated, [Ald97, ABBG12, CG20]).

As $n \to \infty$

$$\left(n^{-1/3} X_{n,i}(\lfloor n^{2/3} t \rfloor); t \geq 0\right)_{i\geq 1} \xrightarrow{d} (e_i(t); t \geq 0)_{i \geq 1}$$

where e_i *are excursions of* $W(t) + \lambda t - \frac{1}{2}t^2$ *above its running minimum re-ordered by length.*

Scaling Limits: Surplus

For the i^{th} largest component, the surplus edges are

$$\text{Bin}\left(A_n(i), n^{-1} + \lambda n^{-4/3}\right)$$

Scaling Limits: Surplus

For the i^{th} largest component, the surplus edges are

$$\mathrm{Bin}\left(A_n(i), n^{-1} + \lambda n^{-4/3}\right)$$

$$\approx \mathrm{Bin}\left(n \int e_i(s)\, ds, n^{-1} + \lambda n^{-4/3}\right)$$

Scaling Limits: Surplus

For the i^{th} largest component, the surplus edges are

$$\text{Bin}\left(A_n(i), n^{-1} + \lambda n^{-4/3}\right)$$

$$\approx \text{Bin}\left(n \int e_i(s)\, ds, n^{-1} + \lambda n^{-4/3}\right) \approx \text{Poisson}\left(\int e_i(s)\, ds\right)$$

Scaling Limits: Surplus

For the i^{th} largest component, the surplus edges are

$$\mathrm{Bin}\left(A_n(i), n^{-1} + \lambda n^{-4/3}\right)$$

$$\approx \mathrm{Bin}\left(n \int e_i(s)\, ds, n^{-1} + \lambda n^{-4/3}\right) \approx \mathrm{Poisson}\left(\int e_i(s)\, ds\right)$$

Theorem 5 (Aldous [Ald97]).

As $n \to \infty$, the largest componets rescale as:

$$\frac{\# C_n(j)}{n^{2/3}} \xrightarrow{d} \zeta_j := \textit{life-time of } e_j.$$

The number of surplus edges of the corresponding component

$$\textit{surplus}(C_n(j)) \xrightarrow{d} \mathrm{Poisson}\left(\int_0^{\zeta_j} e_j(s)\, ds\right)$$

Subsequent Work

Aldous gave a process-level scaling limit for a function which encodes the information of the random graph.

Subsequent Work

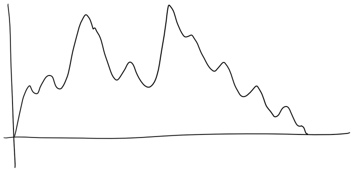
Aldous gave a process-level scaling limit for a function which encodes the information of the random graph.

Theorem 6 (Addario-Berry, Broutin, Goldschmidt [ABBG12]).

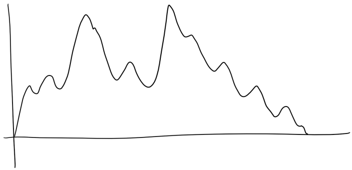
There exists sequence of random metric spaces $(\mathcal{M}_i; i \geq 1)$ such that

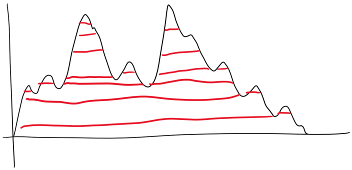
$$\left(n^{-1/3} C_n(i); i \geq 1\right) \xrightarrow{d} (\mathcal{M}_i; i \geq 1)$$

What are the metric spaces?

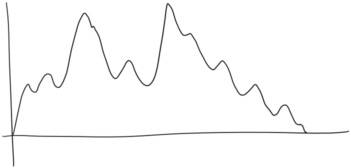


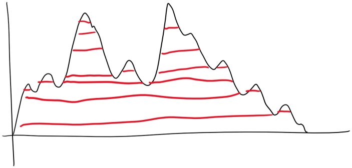
What are the metric spaces?

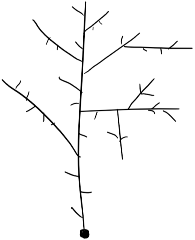




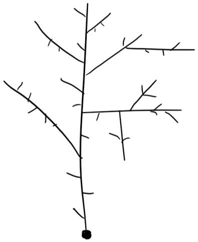
What are the metric spaces?



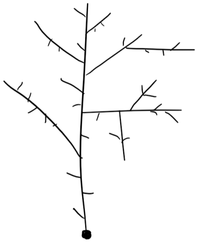


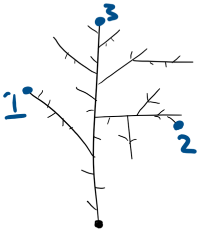


What are the metric spaces?

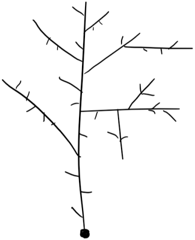


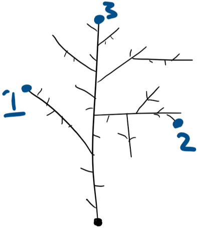
What are the metric spaces?

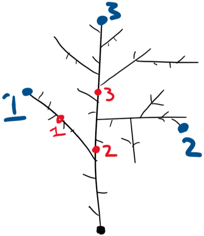




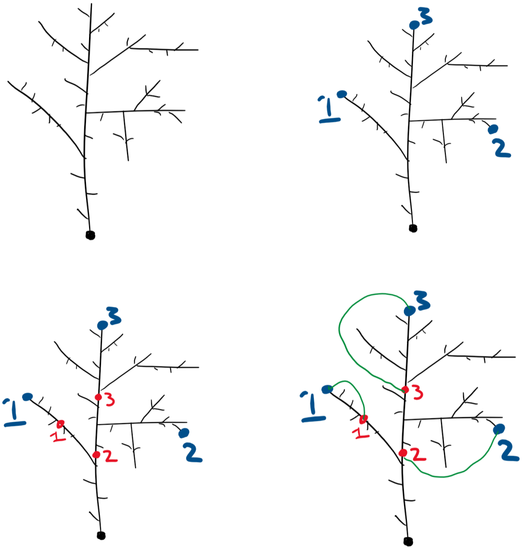
What are the metric spaces?







What are the metric spaces?



An epidemic model

Population of $m + k$ people.

An epidemic model

Population of $m + k$ people.

On day zero k people are infected with a disease, and m people are healthy.

An epidemic model

Population of $m + k$ people.
On day zero k people are infected with a disease, and m people are healthy.
Infected person infects healthy person with probability p on that day, and are forever cured after that day.

An epidemic model

Population of $m + k$ people.
On day zero k people are infected with a disease, and m people are healthy.
Infected person infects healthy person with probability p on that day, and are forever cured after that day.
Then

$$\text{Bin}\,(m, q) \qquad \text{are infected by the next day where}$$

$$q = (1-p)^k = \text{probability of not being infected by the } k \text{ infected people.}$$

This leads to the following Markov chain:

$$Z_n(0) = k, \qquad C_n(0) = k$$

This leads to the following Markov chain:

$$Z_n(0) = k, \qquad C_n(0) = k$$

$$Z_n(h+1) = \begin{array}{l} \#\text{infected people the next day with } Z_n(h) \\ \text{infected people and } n - C_n(h) \text{ healthy people} \end{array}$$

$$C_n(h) = \sum_{j=0}^{h} Z_n(j)$$

This leads to the following Markov chain:

$$Z_n(0) = k, \qquad C_n(0) = k$$

$$Z_n(h+1) = \begin{array}{l}\#\text{infected people the next day with } Z_n(h) \\ \text{infected people and } n - C_n(h) \text{ healthy people}\end{array}$$

$$C_n(h) = \sum_{j=0}^{h} Z_n(j)$$

Theorem 7 (C. [Cla20]).

When $p = n^{-1} + \lambda n^{-4/3}$ and $kn^{-1/3} \to x$ as $n \to \infty$ then

$$\left(n^{-1/3} Z_n(\lfloor n^{1/3} t \rfloor), n^{-2/3} C_n(\lfloor n^{1/3} t \rfloor)\right) \xrightarrow{d} (Z(t), C(t))$$

where (Z, C) solves

$$Z(t) = x + X^\lambda \circ C(t), \quad C(t) = \int_0^t Z(s)\, ds, \quad X^\lambda(t) = W(t) + \lambda t - \frac{1}{2} t^2.$$

Subsequent Works

Generalization of Erdős-Rényi random graph: Rank-1 inhomogeneous model.

Subsequent Works

Generalization of Erdős-Rényi random graph: Rank-1 inhomogeneous model.
Graph on n vertices with edges included

$$\mathbb{P}\left(\{i, j\} \text{ is an edge}\right) = 1 - \exp\left(-q w_i w_j\right),$$

where $w_1 \geq w_2 \geq \cdots \geq w_n > 0$ and some $q \in [0, \infty)$.

Subsequent Works

Generalization of Erdős-Rényi random graph: Rank-1 inhomogeneous model.
Graph on n vertices with edges included

$$\mathbb{P}\left(\{i, j\} \text{ is an edge}\right) = 1 - \exp\left(-q w_i w_j\right),$$

where $w_1 \geq w_2 \geq \cdots \geq w_n > 0$ and some $q \in [0, \infty)$.
Weights are a propensity to have neighbors.

Subsequent Works

Theorem 8 (Aldous, Limic [AL98], Broutin ,Duquesne, Wang [BDW20]).

Under some assumptions (some technical, some natural) the rank-1 inhomogeneous model:

1. *A-L [AL98] a breadth-first walk has a rescaling limit:*

$$\sigma W(t) + \lambda t - \frac{1}{2}\sigma^2 t^2 + \sum_{j \geq 1} \left(c_j 1_{(E_j \leq t)} - c_j^2 t \right),$$

   for a Brownian motion W and some exponential random variables E_j with $\mathbb{E}[E_j] = 1/c_j$.

2. *B-D-W [BDW20]. The components of the model have scaling metric space limits.*

Subsequent Works

A different way to construct random graphs: graphs from degree sequences.

Subsequent Works

A different way to construct random graphs: graphs from degree sequences.
Have n vertices where each vertex has degree $d_j \geq 1$ with $\sum_{j=1}^{n} d_j$ is even.

Subsequent Works

A different way to construct random graphs: graphs from degree sequences.
Have n vertices where each vertex has degree $d_j \geq 1$ with $\sum_{j=1}^{n} d_j$ is even.

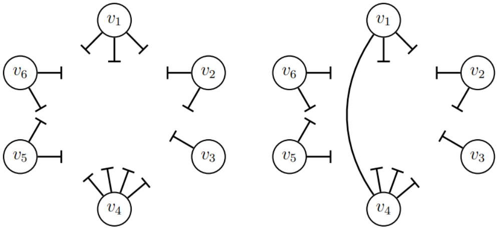


Figure: From [vdH17].

Subsequent Works

Theorem 9 (Vaguely stated below [MR95, MR98], [Jos14], [CG20]).

1. *Molloy, Reed [MR95, MR98]: when the degrees are i.i.d. random variables, there is a phase transition (like for Erdős-Rényi random graphs) where a giant component emerges.*

Subsequent Works

Theorem 9 (Vaguely stated below [MR95, MR98], [Jos14], [CG20]).

1. *Molloy, Reed [MR95, MR98]: when the degrees are i.i.d. random variables, there is a phase transition (like for Erdős-Rényi random graphs) where a giant component emerges.*
2. *Joseph [Jos14]: For D_j i.i.d. with power law distribution, there is an encoding walk with scaling limit as*

$$\alpha\text{-stable analog of} \quad W(t) - \frac{1}{2}t^2.$$

Subsequent Works

Theorem 9 (Vaguely stated below [MR95, MR98], [Jos14], [CG20]).

1. *Molloy, Reed [MR95, MR98]: when the degrees are i.i.d. random variables, there is a phase transition (like for Erdős-Rényi random graphs) where a giant component emerges.*
2. *Joseph [Jos14]: For D_j i.i.d. with power law distribution, there is an encoding walk with scaling limit as*

$$\alpha\text{-stable analog of} \quad W(t) - \frac{1}{2}t^2.$$

3. *Conchon-Kerjan, Goldschmidt [CG20]: There exists metric space scaling limits which are the α-stable versions of the "Brownian graphs" in the Erdős-Rényi case.*

L. Addario-Berry, N. Broutin, and C. Goldschmidt.
The continuum limit of critical random graphs.
Probab. Theory Related Fields, 152(3-4):367–406, 2012.

David Aldous and Vlada Limic.
The entrance boundary of the multiplicative coalescent.
Electron. J. Probab., 3:No. 3, 59 pp. 1998.

David Aldous.
Brownian excursions, critical random graphs and the multiplicative coalescent.
Ann. Probab., 25(2):812–854, 1997.

Nicolas Broutin, Thomas Duquesne, and Minmin Wang.
Limits of multiplicative inhomogeneous random graphs and Lévy trees: Limit theorems.
arXiv e-prints, page arXiv:2002.02769, February 2020.

Béla Bollobás.
The evolution of random graphs.
Trans. Amer. Math. Soc., 286(1):257–274, 1984.

Guillaume Conchon–Kerjan and Christina Goldschmidt.
The stable graph: the metric space scaling limit of a critical random graph with i.i.d. power-law degrees.
arXiv e-prints, page arXiv:2002.04954, February 2020.

David Clancy, Jr.
A new relationship between Erdős-Rényi graphs, epidemic models and Brownian motion with parabolic drift.
arXiv e-prints, page arXiv:2006.06838, June 2020.

P. Erdős and A. Rényi.
On the evolution of random graphs.
Magyar Tud. Akad. Mat. Kutató Int. Közl., 5:17–61, 1960.

Adrien Joseph.
The component sizes of a critical random graph with given degree sequence.
Ann. Appl. Probab., 24(6):2560–2594, 2014.

Tomasz Łuczak, Boris Pittel, and John C. Wierman.
The structure of a random graph at the point of the phase transition.
Trans. Amer. Math. Soc., 341(2):721–748, 1994.

Michael Molloy and Bruce Reed.
A critical point for random graphs with a given degree sequence.
In *Proceedings of the Sixth International Seminar on Random Graphs and Probabilistic Methods in Combinatorics and Computer Science, "Random Graphs '93" (Poznań, 1993)*, volume 6, pages 161–179, 1995.

Michael Molloy and Bruce Reed.
The size of the giant component of a random graph with a given degree sequence.
Combin. Probab. Comput., 7(3):295–305, 1998.

Joel Spencer.
Enumerating graphs and Brownian motion.
Comm. Pure Appl. Math., 50(3):291–294, 1997.

Remco van der Hofstad.
Random graphs and complex networks. Vol. 1.
Cambridge Series in Statistical and Probabilistic Mathematics, [43].
Cambridge University Press, Cambridge, 2017.